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A connection between quantum Hilbert-space and classical phase-space operators

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Abstract. We develop a general 'classicalization' procedure that links Hilbert-space and phase-space operators, using Weyl's operator. Then we transform the time-dependent Schrödinger equation into a phase-space picture using free parameters. They include position Q and momentum P. We expand the phase-space Hamiltonian in an $\hbar$-Taylor series and fix parameters with the condition that coefficients of $\hbar^0$, $-i\hbar^1\,\partial/\partial Q$ and $i\hbar^1\,\partial/\partial P$ vanish. This condition results in *generalized Hamilton equations* and a natural link between classical and quantum dynamics, while the quantum motion-equation remains *exact*. In this picture, the Schrödinger equation reduces in the classical limit to a generalized Liouville equation for the quantum-mechanical system state. We modify Glauber's coherent states with a suitable phase factor $S(Q, P, t)$ and use them to obtain phase-space representations of quantum dynamics and quantum-mechanical quantities.

(Some figures in this article are in colour only in the electronic version; see www.iop.org)

1. Introduction

Many authors have investigated the reformulation of Hilbert-space quantum mechanics in terms of classical phase space. The Weyl–Wigner–Moyal formalism associates a quantum state in Hilbert-space with a real-valued function called the Wigner function [1–4]. This function is only partially equivalent to classical distribution functions as it can assume negative values. Although the Wigner function has proven useful to determine average values for a large class of ordinary momentum and position functions, for some cases it gives incorrect results [5, 6].

Husimi [7] introduced a procedure to smooth the Wigner function using a Gaussian. It led to the so-called Husimi or antinormal ordering function. Although this distribution function is always positive, occasionally the operator's phase-space c-number representatives may not be well defined [4, 8].

While the Weyl–Wigner–Moyal and Husimi methods describe quantum mechanics in terms of density operators, in this paper we will use state-vectors. We follow a more conventional approach to relate different pictures to each other (e.g. Schrödinger, Heisenberg, interaction), using quantum-mechanical transformation theory. Therefore, we introduce a *phase-space picture* for the Schrödinger equation to explore the relationship between classical and quantum dynamics (section 5). We will find that for a given quantum system the corresponding classical system obeys the generalized Hamilton equations.

The basis of the proposed method is a general *'classicalization' procedure* deduced in the first part of this paper (sections 2–4). It directly connects a given function $F(\hat{q}, \hat{p}, t)$

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of *non-commuting* position and momentum operators, $(\hat{q}, \hat{p})$, with a corresponding phase-space function $F(Q, P, \breve{Q}, \breve{P}, t)$ of suitable *classical* entities, $(Q, P, \breve{Q}, \breve{P}, t)$. To deduce the 'classicalization' method, we avoid using representation theory. Coherent-state representation will only be used in the last part (section 6) where we replace kets by phase-space wavefunctions.

There are several approaches for the 'classicalization', that is, for the problem of constructing the classical counterpart of a quantum-mechanical system, and for the study of the way in which the classical description can be obtained as a limiting form of the quantum-mechanical one. The simplest connection between quantum and classical mechanics goes back to Ehrenfest [9]. One approach, as investigated by Heslot [10], is to rewrite the Schrödinger equation as a set of Hamilton equations using as 'coordinates' and 'momenta' the real and imaginary parts of the expansion coefficients of the wavefunction over an orthonormal basis. Other treatments [11, 12] establish that a quantum-mechanical system is exactly equivalent to a large classical system that consists of two parts: the ordinary classical analogue of the original quantum system and a subsystem consisting of an infinite number of additional classical degrees of freedom. The coherent-state path-integral formalism and the stationary phase approximation also lead to classical-like equations [13].

Another systematic approach to associating quantum mechanics with a classical system is the so-called semiquantal dynamics, or squeezed state dynamics, which starts with the variational restriction of the Schrödinger equation to a subspace of the full Hilbert-space, uses generalized coherent states as a trial wavefunction, and obtains canonical equations of motion for expectation values and quantum fluctuations [14–17]. The standard Heller semiclassical dynamics of Gaussian wavepackets [18] arises as consistent truncations (in $\hbar$) to semiquantal dynamics.

2. Relationship between Hilbert-space and phase-space operators

Let us consider a quantum-mechanical system with f degrees of freedom and *Hilbert-space* $\mathcal{H}$. Identity, position and momentum operators are denoted by $\hat{1}$, $\hat{q} = \{\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_f\}$ and $\hat{p} = \{\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_f\}$. They satisfy canonical commutation relations, $[\hat{q}_n, \hat{p}_m] = \mathrm{i}\hbar\,\hat{1}\,\delta_{nm}$, for $n, m = 1, 2, \ldots, f$. Henceforth the Hilbert-space operators acting in *Hilbert-space* $\mathcal{H}$ are denoted by a caret ($\hat{}$).

In $\mathcal{H}$ we define the phase-space (or Weyl) displacement operator $\hat{D}(Q, P)$, and its inverse $\hat{D}(-Q, -P)$, that can be decomposed as

$$\hat{D}(-Q, -P) := \exp\left(-\frac{\mathrm{i}}{\hbar}\left(P\hat{q} - Q\hat{p}\right)\right) \tag{1a}$$

$$= \underset{-}{w}(Q, P/2)\,\underset{+}{\hat{D}}(Q, P) = \underset{+}{w}(Q, P/2)\,\underset{-}{\hat{D}}(Q, P). \tag{1b}$$

Here we use auxiliary operators,

$$\underset{+}{\hat{D}}(Q, P) := \exp\left(-\frac{\mathrm{i}}{\hbar}P\hat{q}\right)\exp\left(\frac{\mathrm{i}}{\hbar}Q\hat{p}\right) \tag{2a}$$

$$\underset{-}{\hat{D}}(Q, P) := \exp\left(\frac{\mathrm{i}}{\hbar}Q\hat{p}\right)\exp\left(-\frac{\mathrm{i}}{\hbar}P\hat{q}\right) \tag{2b}$$

and phase factors

$$\underset{\pm}{w}(Q, P) := \exp\left(\pm\frac{\mathrm{i}}{\hbar}QP\right). \tag{3}$$

In terms of equations (1*a*), (1*b*) and (2*a*), (2*b*), the adjoint equations are

$$\left(\underset{\pm}{\hat{D}}(Q,P)\right)^{\dagger} = \underset{\mp}{\hat{D}}(-Q^*,-P^*) \qquad \left(\hat{D}(-Q,-P)\right)^{\dagger} = \hat{D}(Q^*,P^*) \tag{4}$$

where the asterisk (*) indicates complex conjugation. The Weyl operator depends on coordinates and momenta, $Q = (Q_1, Q_2, \ldots, Q_f)$ and $P = (P_1, P_2, \ldots, P_f)$, that are free parameters and may be chosen as real or complex quantities in phase space Ω. If Q and P are real (Q, P) can be regarded as points in the $2f$-dimensional real phase space $(\Omega = R^{2f})$. We can simplify the notation in this case by defining $w(Q,P) := \underset{+}{w}(Q,P)$ and $w^*(Q,P) := \underset{-}{w}(Q,P)$.

Operators acting in phase space Ω will be denoted by a circumflex ($\breve{}$). Translations in classical phase space Ω can be generated with classical (commuting) operators, $\breve{Q} = (\breve{Q}_1, \breve{Q}_2, \ldots, \breve{Q}_f)$ and $\breve{P} = (\breve{P}_1, \breve{P}_2, \ldots, \breve{P}_f)$, that act on the space of smooth functions on Ω and are defined by

$$\breve{Q}_n := \mathrm{i}\hbar\partial/\partial P_n \qquad \breve{P}_n := -\mathrm{i}\hbar\partial/\partial Q_n \qquad \left[\breve{Q}_n, \breve{P}_m\right] = 0 \qquad n, m = 1, 2, \ldots, f. \tag{5}$$

Now, for every function $G(Q,P) := G(Q,P;\hat{q},\hat{p})$ and f-dimensional vectors a and b (a and b independent of Q and P) we have

$$\exp\left(-\frac{\mathrm{i}}{\hbar}b\breve{Q}\right)\exp\left(\frac{\mathrm{i}}{\hbar}a\breve{P}\right)G(Q,P) = G(Q+a, P+b) \tag{6}$$

which implies

$$\exp\left(\frac{\mathrm{i}}{\hbar}a\breve{P}\right)\underset{+}{\hat{D}}(Q,P) = \underset{+}{\hat{D}}(Q,P)\exp\left(\frac{\mathrm{i}}{\hbar}a\hat{p}\right) \tag{7a}$$

$$\exp\left(-\frac{\mathrm{i}}{\hbar}a\breve{Q}\right)\underset{-}{\hat{D}}(Q,P) = \underset{-}{\hat{D}}(Q,P)\exp\left(-\frac{\mathrm{i}}{\hbar}a\hat{q}\right). \tag{7b}$$

In quantum mechanics we deal with functions of non-commuting operators $\hat{q}$ and $\hat{p}$. Now we use the above identities to establish formal relationships between a given function of *non-commuting* Hilbert operators $(\hat{q}, \hat{p})$ and the corresponding function of *classical* entities $(Q, P, \breve{Q}, \breve{P})$. For this purpose we do not use representation theory and avoid the so-called ordering problem [20].

We begin by introducing a set of $(N+1)$ phase-space functions

$$\left\{\overset{N}{F}(Q), \overset{N-1}{F}(P), \ldots, \overset{2}{F}(P), \overset{1}{F}(Q), \overset{0}{F}(P)\right\}$$

labelled with numbers on top. We assume that each function is expandable in a convergent power series in Q or P. By *convention*, functions of Q are odd numbered and functions of P are even numbered. We now define Hilbert operators, using the substitutions $Q \to \hat{q}$ and $P \to \hat{p}$. They set up well defined bi-directional mappings that relate phase-space functions to Hilbert-space operators:

$$\overset{2k+1}{F}(\hat{q}) \leftrightarrow \overset{2k+1}{F}(Q) \qquad \overset{2k}{F}(\hat{p}) \leftrightarrow \overset{2k}{F}(P).$$

Let us consider a general Hilbert-space operators constructed as products of functions of $\hat{q}$ and functions of $\hat{p}$, namely

$$\underset{+}{\overset{N}{F}}(\hat{q},\hat{p}) := \begin{cases} \overset{N}{F}(\hat{p})\ldots\overset{2}{F}(\hat{p})\,\overset{1}{F}(\hat{q})\,\overset{0}{F}(\hat{p}) & N \text{ even} \\ \overset{N}{F}(\hat{q})\ldots\overset{2}{F}(\hat{p})\,\overset{1}{F}(\hat{q})\,\overset{0}{F}(\hat{p}) & N \text{ odd} \end{cases} \tag{8a}$$

and

$$\overset{N}{\underset{-}{F}}(\hat{q},\hat{p}) := \begin{cases} \overset{0}{F}(\hat{p})\,\overset{1}{F}(\hat{q})\,\overset{2}{F}(\hat{p})\ldots\overset{N}{F}(\hat{p}) & N \text{ even} \\ \overset{0}{F}(\hat{p})\,\overset{1}{F}(\hat{q})\,\overset{2}{F}(\hat{p})\ldots\overset{N}{F}(\hat{q}) & N \text{ odd.} \end{cases} \tag{8b}$$

We also consider general Hilbert-space operators constructed as

$$\underset{\pm}{F}(\hat{q},\hat{p}) = \sum_{N=0}^{\infty}\left[\overset{2N}{\underset{\pm}{C}}\,\overset{2N}{\underset{\pm}{F}}(\hat{q},\hat{p}) + \overset{2N+1}{\underset{\pm}{C}}\,\overset{2N+1}{\underset{\pm}{F}}(\hat{q},\hat{p})\right] \tag{9}$$

where coefficients $\overset{k}{\underset{\pm}{C}}$ are real or complex quantities. Note that operators (8*a*) and (8*b*) have been defined as an ordered product of functions of $\hat{q}$ and $\hat{p}$, without being ordered operators (all $\hat{q}$s to the left of all $\hat{p}$s or vice versa).

Equations (7*a*) and (7*b*) are relevant as they enable us to deduce the following *intertwining relations* for Hilbert-space and phase-space operators (see appendix A):

$$\underset{+}{F}(Q,P,\breve{Q},\breve{P})\,\underset{+}{\hat{D}}(Q,P) = \underset{+}{\hat{D}}(Q,P)\,\underset{+}{F}(\hat{q},\hat{p}) \tag{10a}$$

$$\underset{-}{F}(Q,P,\breve{Q},\breve{P})\,\underset{-}{\hat{D}}(Q,P) = \underset{-}{F}(\hat{q},\hat{p})\,\underset{-}{\hat{D}}(Q,P). \tag{10b}$$

The essential role of operators $\underset{\pm}{\hat{D}}(Q,P)$ is to connect Hilbert-space operators $\underset{\pm}{F}(\hat{q},\hat{p})$ with corresponding phase-space operators $\underset{\pm}{F}(Q,P,\breve{Q},\breve{P})$ and vice versa. Note that entities as $\underset{\pm}{\hat{D}}(Q,P)$ play a double role:

(a) they act as Hilbert-space operators, since they depend of $\hat{q}$ and $\hat{p}$;
(b) they are also non-scalar functions in phase space Ω, because they are Q- and P dependent, but $\hat{q}$ and $\hat{p}$ obey a non-commutative algebra.

The operators $\underset{\pm}{F}(Q,P,\breve{Q},\breve{P})$ can act in both spaces ($\mathcal{H}$ and Ω).

Now *we need* to obtain *explicit expressions* for phase-space operators $\underset{\pm}{F}(Q,P,\breve{Q},\breve{P})$ associated with (8*a*) and (8*b*). We start with equations (see appendix A)

$$\overset{2k}{\underset{-}{F}}(Q,P,\breve{P})\,\underset{+}{\hat{D}}(Q,P) = \underset{-}{\hat{D}}(Q,P)\,\overset{2k}{F}(\hat{p}) \tag{11a}$$

$$\overset{2k+1}{\underset{+}{F}}(Q,P,\breve{Q})\,\underset{-}{\hat{D}}(Q,P) = \underset{+}{\hat{D}}(Q,P)\,\overset{2k+1}{F}(\hat{q}). \tag{11b}$$

$$\overset{2k}{\underset{+}{F}}(Q,P,\breve{P})\,\underset{-}{\hat{D}}(Q,P) = \overset{2k}{F}\left(\hat{p}\right)\underset{+}{\hat{D}}(Q,P) \tag{11c}$$

$$\overset{2k+1}{\underset{-}{F}}(Q,P,\breve{Q})\,\underset{-}{\hat{D}}(Q,P) = \overset{2k+1}{F}\left(\hat{q}\right)\underset{-}{\hat{D}}(Q,P) \tag{11d}$$

Combining equations (8*a*) and (8*b*) and (11*a*) and (11*b*) we obtain

$$\overset{N}{\underset{+}{F}}(Q,P,\breve{Q},\breve{P}) := \begin{cases} \underset{+}{w}(Q,P)\,\overset{N}{\underset{-}{F}}(Q,P,\breve{P})\ldots\overset{2}{\underset{-}{F}}(Q,P,\breve{P}) & \\ \qquad\times\,\overset{1}{\underset{+}{F}}(Q,P,\breve{Q})\,\overset{0}{\underset{-}{F}}(Q,P,\breve{P}) & N \text{ even} \\ \overset{N}{\underset{+}{F}}(Q,P,\breve{Q})\ldots\overset{2}{\underset{-}{F}}(Q,P,\breve{P}) & \\ \qquad\times\,\overset{1}{\underset{+}{F}}(Q,P,\breve{Q})\,\overset{0}{\underset{-}{F}}(Q,P,\breve{P}) & N \text{ odd} \end{cases} \tag{12a}$$

and

$$\overset{N}{\underset{-}{F}}(Q,P,\breve{Q},\breve{P}) := \begin{cases} \underset{-}{w}(Q,P)\,\overset{N}{\underset{+}{F}}(Q,P,\breve{P})\ldots\overset{2}{\underset{+}{F}}(Q,P,\breve{P}) \\ \quad\times\overset{1}{\underset{-}{F}}(Q,P,\breve{Q})\,\overset{0}{\underset{+}{F}}(Q,P,\breve{P}) & N \text{ even} \\ \overset{N}{\underset{-}{F}}(Q,P,\breve{Q})\ldots\overset{2}{\underset{+}{F}}(Q,P,\breve{P}) \\ \quad\times\overset{1}{\underset{-}{F}}(Q,P,\breve{Q})\,\overset{0}{\underset{+}{F}}(Q,P,\breve{P}) & N \text{ odd} \end{cases} \tag{12b}$$

where we have introduced the *notation*

$$\overset{2k+1}{\underset{\pm}{F}}(Q,P,\breve{Q}) := \underset{\pm}{w}(Q,P)\,\overset{2k+1}{F}(\breve{Q}) \qquad \overset{2k}{\underset{\pm}{F}}(Q,P,\breve{P}) := \underset{\pm}{w}(Q,P)\,\overset{2k}{F}(\breve{P}). \tag{12c}$$

In order to understand the fundamental relations (11*a*) and (11*b*) better, we recommend the reader compares (8*a*) with (12*a*) and (8*b*) with (12*b*). In the first case, the action order of Hilbert-space and phase-space operators is the same (see the number over the functions). In the second case, the action order is inverted. Also note that although $\breve{Q}$ and $\breve{P}$ are phase-space commuting operators, in (12*a*) and (12*b*) the order of the factors must be preserved due to exponential functions $\underset{\pm}{w}(Q,P)$. Observe that for even N extra factors $\underset{+}{w}(Q,P)$ and $\underset{-}{w}(Q,P)$ are present at the right-hand side of (12*a*) and (12*b*). These factors allow equations (10*a*) and (10*b*) to be correct independently of the value of N (even or odd) and, therefore, equations (10*a*) and (10*b*) are valid for general Hilbert-space operator (9).

3. Classical–quantum correspondence between operators

We call *'classicalization'* the procedure that assigns to a Hilbert-space operator corresponding classical functions (*symbols*). We could derive an infinite set of 'classicalization' schemes from phase-space operators (12*a*) and (12*b*). To avoid the difference in (12*a*) and (12*b*) arising from even and odd Ns, we introduce an *arbitrary smooth phase-space function* $S(Q,P,t)$ and define phase-space operators

$$\begin{aligned}\breve{Q}_{\pm}[S] &:= \exp\left[\pm\frac{\mathrm{i}}{\hbar}(QP - S(Q,P,t))\right]\breve{Q}\exp\left[\mp\frac{\mathrm{i}}{\hbar}(QP - S(Q,P,t))\right] \\ &= \pm q(Q,P,t) + \breve{Q}\end{aligned} \tag{13a}$$

and

$$\begin{aligned}\breve{P}_{\pm}[S] &:= \exp\left[\mp\frac{\mathrm{i}}{\hbar}S(Q,P,t)\right]\breve{P}\exp\left[\pm\frac{\mathrm{i}}{\hbar}S(Q,P,t)\right] \\ &= \pm p(Q,P,t) + \breve{P}\end{aligned} \tag{13b}$$

where we have introduced for convenience *auxiliary functions*

$$q(Q,P,t) := Q - \partial S(Q,P,t)/\partial P \qquad p(Q,P,t) := \partial S(Q,P,t)/\partial Q. \tag{14}$$

Phase-space operators $\breve{Q}_{\pm}[S]$ and $\pm\breve{P}_{\pm}[S]$ are canonical conjugate, because they satisfy $\pm\left[\breve{Q}_{\pm}[S],\breve{P}_{\pm}[S]\right] = \mathrm{i}\hbar\hat{1}$. $\breve{Q}$ and $\breve{P}$ are commuting operators, i.e. $\left[\breve{Q},\breve{P}\right] = 0$. To simplify the notation we write from now on q, p, $\breve{Q}_{\pm}$, and $\breve{P}_{\pm}$ instead of $q(Q,P,t)$, $p(Q,P,t)$, $\breve{Q}_{\pm}[S]$ and $\breve{P}_{\pm}[S]$.

We define the transformation

$$\overset{N}{\underset{\pm}{F}}\left(\breve{Q}_{\pm},\breve{P}_{\pm},t\right) := \exp\left[\mp\frac{\mathrm{i}}{\hbar}S(Q,P,t)\right]\overset{N}{\underset{\pm}{F}}\left(Q,P,\breve{Q},\breve{P},t\right)\exp\left[\pm\frac{\mathrm{i}}{\hbar}S(Q,P,t)\right] \tag{15}$$

where Hilbert-space operators and function $S(Q, P, t)$ can depend explicitly on time t. Then, using (12a) and (12b) and (13a) and (13b), we find for (15) the simple expression

$$\overset{N}{\underset{\pm}{\mathcal{F}}}\big(\breve{Q}_\pm, \breve{P}_\pm, t\big) = \begin{cases} \overset{N}{F}\big(\breve{P}_\pm, t\big) \ldots \overset{2}{F}\big(\breve{P}_\pm, t\big)\,\overset{1}{F}\big(\breve{Q}_\pm, t\big)\,\overset{0}{F}\big(\breve{P}_\pm, t\big) & N \text{ even} \\ \overset{N}{F}\big(\breve{Q}_\pm, t\big) \ldots \overset{2}{F}\big(\breve{P}_\pm, t\big)\,\overset{1}{F}\big(\breve{Q}_\pm, t\big)\,\overset{0}{F}\big(\breve{P}_\pm, t\big) & N \text{ odd.} \end{cases} \tag{16}$$

Note that (16) can be derived from (8a) and (8b) by replacing operators $\hat{q}$ and $\hat{p}$ with phase-space equivalents $\breve{Q}_\pm$ and $\breve{P}_\pm$ (i.e. $\hat{q} \leftrightarrow \breve{Q}_\pm$ and $\hat{p} \leftrightarrow \breve{P}_\pm$) and by reversing the order of the operator product for those with subscript minus (−) (compare (8b) and (16)). Different 'classicalization' schemes may now be constructed by suitable choices of $S(Q, P, t)$, since $S(Q, P, t)$ is arbitrary.

4. Expansion of phase-space operators

In this section we derive an expansion for a function of $\breve{Q}_+$ or $\breve{P}_+$, and for products of operators, like those at the right-hand side of (16). The starting point is the Taylor expansion

$$F(y + x) = \exp(x\partial/\partial y)\,F(y) = \sum_{\nu=0}^{\infty} F_\nu(y)\,x^\nu \tag{17a}$$

where x and y are commuting entities, and the coefficients are given by

$$F_\nu(y) := \frac{1}{\nu!}\,(\partial/\partial y)^\nu\,F(y). \tag{17b}$$

Note we are using a multi-index notation, where ν is an ordered set $(\nu_1, \nu_2, \ldots, \nu_f)$, consisting of f non-negative integers restricted by $\nu = \nu_1 + \nu_2 + \cdots + \nu_f$ and $\nu! = \nu_1!\nu_2!\ldots\nu_f!$.

4.1. Functions of $\breve{Q}_+$ or $\breve{P}_+$

Even without knowing the form of function $q(Q, P, t)$ in (14), we may decompose it as $q = \overline{q} + X$, where $\overline{q} = \overline{q}(Q, t)$ is a P-independent function, and $X(Q, P, t) := q - \overline{q}$. Similarly, $p = \overline{p} + Y$, where $\overline{p} = \overline{p}(P, t)$ is a Q-independent function, and $Y(Q, P, t) := p - \overline{p}$. Thus, commutation relations $[\pm\overline{q}, \breve{Q}] = [\pm\overline{p}, \breve{P}] = 0$ allow us to express phase-space operators present in (16) as

$$\overset{2k}{F}\big(\breve{P}_\pm, t\big) = \overset{2k}{F}\big(\pm p + \breve{P}, t\big) = \sum_{m=0}^{\infty} \overset{2k}{F}_m\big(\pm\overline{p}, t\big)\big(\pm Y + \breve{P}\big)^m \tag{18}$$

$$\overset{2k+1}{F}\big(\breve{Q}_\pm, t\big) = \overset{2k+1}{F}\big(\pm q + \breve{Q}, t\big) = \sum_{n=0}^{\infty} \overset{2k+1}{F}_n\,(\pm\overline{q}, t)\,\big(\pm X + \breve{Q}\big)^n. \tag{19}$$

Hereafter, and for the purposes of the present paper, it will be sufficient to consider only functions of operators $\breve{Q}_+$ and $\breve{P}_+$.

We proceed now to evaluate $\big(X + \breve{Q}\big)^n$. First, note that

$$\breve{Q}X = X\breve{Q} + (\breve{Q}X) \qquad \breve{P}Y = Y\breve{P} + (\breve{P}Y) \tag{20}$$

where the notation $(\breve{Q}X)$ indicates that the operator $\breve{Q}$ acts only inside the parentheses; that is, the whole entity $(\breve{Q}X) = \mathrm{i}\hbar\partial X/\partial P$ is a multiplication operator. Because of (20), for every positive number $n \geqslant 0$ we get the *generalized binomial expansion*

$$\big(X + \breve{Q}\big)^n = \sum_{\mu=0}^{n} X_{n\mu}\,\breve{Q}^\mu. \tag{21}$$

The coefficients $\{X_{n\mu}(Q, P, t), \mu = 0, 1, 2, \ldots, n\}$ are functions of (Q, P, t), and we see that $X_{00} = 1$, $X_{10} = X$, and $X_{11} = 1$.

At the left-hand side of (21) we write $(\cdots)^n$ as $(\cdots)(\cdots)^{n-1}$, use (21) to expand $(\cdots)^{n-1}$, employ the first equation (20) in the form $\breve{Q}X_{n-1,\mu} = X_{n-1,\mu}\breve{Q} + (\breve{Q}X_{n-1,\mu})$, and put a sum into the normal form (21) by means of a change of indices. In this way we find the recurrence relation

$$X_{n,\mu} = X_{n-1,\mu-1} + XX_{n-1,\mu} + (\breve{Q}X_{n-1,\mu}) \tag{22}$$

where, by convention, $X_{n,\mu} = 0$ if $n \geqslant 0$ and $\mu < 0$, or $n < \mu$. Moreover, we have $X_{n,n} = 1$, for $n \geqslant 0$, and $X_{10} = X$. In particular, using (22) we obtain

$$X_{n0} = X^n + \sum_{\ell=0}^{n-2}(\breve{Q}X_{n-\ell-1,0})X^\ell \qquad n \geqslant 2. \tag{23}$$

Appendix B includes some coefficients $X_{n\mu}$ obtained by application of (23) and (22).

Inserting (21) into (19), and by using the identity

$$\sum_{n=0}^{\infty}\sum_{\mu=0}^{n} A(n, \mu) = \sum_{\mu=0}^{\infty}\sum_{n=0}^{\infty} A(n+\mu, \mu) \tag{24}$$

we obtain the expansion

$$\overset{2k+1}{F}\left(\breve{Q}_+, t\right) = \sum_{\mu=0}^{\infty} \overset{2k+1}{\mathcal{F}_\mu}(Q, P, t)\breve{Q}^\mu \tag{25}$$

where the coefficients are functions given by

$$\overset{2k+1}{\mathcal{F}_\mu}(Q, P, t) := \sum_{n=0}^{\infty} \overset{2k+1}{F_{n+\mu}}\left(\overline{q}, t\right)X_{n+\mu,\mu}(Q, P, t). \tag{26}$$

Quite analogously, and by noting (20), we establish for (18) the expansion

$$\overset{2k}{F}\left(\breve{P}_+, t\right) = \sum_{\nu=0}^{\infty} \overset{2k}{\mathcal{F}_\nu}(Q, P, t)\breve{P}^\nu \tag{27}$$

with coefficients

$$\overset{2k}{\mathcal{F}_\nu}(Q, P, t) := \sum_{m=0}^{\infty} \overset{2k}{F}_{m+\nu}\left(\overline{p}, t\right)Y_{m+\nu,\nu}(Q, P, t). \tag{28}$$

Coefficients $Y_{m\nu}$ obey equations similar to (23) and (22), but $X \to Y$ and $\breve{Q} \to \breve{P}$.

In the next subsection we will consider expansions for an operator product, such as those in (16). For this we need the Leibnitz rule,

$$\breve{P}^n Y = \sum_{\nu=0}^{n} \binom{n}{\nu}(\breve{P}^{n-\nu}Y)\breve{P}^\nu \tag{29}$$

and a similar expression for $\breve{Q}^n X$. Note that (29) is obtained by using (20) repeatedly.

4.2. Generic form of a generalized Hamiltonian

As an *example* of (9), consider a quite general Hamiltonian

$$\hat{H}(t) := H(\hat{q}, \hat{p}, t) = \overset{0}{F}(\hat{p}, t) + \overset{1}{F}(\hat{q}, t) + \overset{2}{F}(\hat{p}, t)\overset{3}{F}(\hat{q}, t) + \overset{5}{F}(\hat{q}, t)\overset{4}{F}(\hat{p}, t). \tag{30}$$

Here, t is the time, and $\overset{2k+1}{F}(\hat{q}, t)$ and $\overset{2k}{F}(\hat{p}, t)$ are arbitrary functions of all position and momentum operators, $\hat{q}$ and $\hat{p}$. (30) includes the *standard Hamiltonian* (kinetic energy plus potential energy) as a particular case. It allows us, thus, to present the most relevant properties of the herein proposed method. According to (16), and for an arbitrary smooth function $S(Q, P, t)$, the phase-space operator associated with $\hat{H}(t) := H(\hat{q}, \hat{p}, t)$ is given by

$$\underset{+}{\mathcal{H}}\left(\breve{Q}_+, \breve{P}_+, t\right) = \overset{0}{F}(\breve{P}_+, t) + \overset{1}{F}(\breve{Q}_+, t) + \overset{2}{F}(\breve{P}_+, t)\, \overset{3}{F}(\breve{Q}_+, t) + \overset{5}{F}(\breve{Q}_+, t)\, \overset{4}{F}(\breve{P}_+, t). \tag{31}$$

By employing (25), (27), (29) and (24), we get the Taylor expansions

$$\overset{2}{F}\left(\breve{P}_+, t\right) \overset{3}{F}\left(\breve{Q}_+, t\right) = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \overset{2,3}{\mathcal{F}}_{\nu,\mu}(Q, P, t)\, \breve{P}^{\nu} \breve{Q}^{\mu} \tag{32}$$

$$\overset{5}{F}\left(\breve{Q}_+, t\right) \overset{4}{F}\left(\breve{P}_+, t\right) = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \overset{5,4}{\mathcal{F}}_{\nu,\mu}(Q, P, t) \breve{P}^{\nu} \breve{Q}^{\mu}. \tag{33}$$

Here, the coefficients are given by

$$\overset{2,3}{\mathcal{F}}_{\nu,\mu}(Q, P, t) = \sum_{n=0}^{\infty} \binom{n+\nu}{\nu} \overset{2}{\mathcal{F}}_{n+\nu}(Q, P, t)\big(\breve{P}^n \overset{3}{\mathcal{F}}_{\mu}(Q, P, t)\big) \tag{34}$$

and

$$\overset{5,4}{\mathcal{F}}_{\nu,\mu}(Q, P, t) = \sum_{n=0}^{\infty} \binom{n+\mu}{\mu} \overset{5}{\mathcal{F}}_{n+\mu}(Q, P, t)\big(\breve{Q}^n \overset{4}{\mathcal{F}}_{\nu}(Q, P, t)\big). \tag{35}$$

Inserting (25), (27), (32), and (33) into (31) gives

$$\begin{aligned}\underset{+}{\mathcal{H}}\left(\breve{Q}_+, \breve{P}_+, t\right) = {} & \sum_{\nu=0}^{\infty} \overset{0}{\mathcal{F}}_{\nu}(Q, P, t) \breve{P}^{\nu} + \sum_{\mu=0}^{\infty} \overset{1}{\mathcal{F}}_{\mu}(Q, P, t) \breve{Q}^{\mu} \\ & + \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \left[\overset{2,3}{\mathcal{F}}_{\nu,\mu}(Q, P, t) + \overset{5,4}{\mathcal{F}}_{\nu,\mu}(Q, P, t)\right] \breve{P}^{\nu} \breve{Q}^{\mu}. \end{aligned} \tag{36}$$

By a simple extension of the above results, for an arbitrary smooth phase-space function $S(Q, P, t)$ and a Hamiltonian $\hat{H} = H(\hat{q}, \hat{p}, t)$ of the form (9), we may assume the existence of a phase-space representative (see (8*a*) and (16))

$$\underset{+}{\mathcal{H}}\left(\breve{Q}_+, \breve{P}_+, t\right) = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \left[\frac{1}{\nu!\mu!} K_{\nu\mu}(Q, P, t)\right] \breve{P}^{\nu} \breve{Q}^{\mu} \tag{37}$$

with suitable coefficients $K_{\nu\mu}(Q, P, t)$. The phase-space function $K(Q, P, t) := K_{00}(Q, P, t)$ can be interpreted as a *generalized Hamiltonian* associated with the *classical system* that underlies, or that corresponds to, the quantum system described by Hilbert-space Hamiltonian $\hat{H} = H(\hat{q}, \hat{p}, t)$ (see (9) or (31)). $K(Q, P, t)$ is a function of coordinates Q and momenta P, and it depends parametrically on time t. Note that all of the above relations are exact and that we are not dealing with any semiclassical approximations ($\hbar \to 0$).

4.3. A particular choice of $S(Q, P, t)$

Let us consider a *particular* phase-space function, $S(Q, P, t) = -s(t) - x(t)P + y(t)Q + a(t)QP$, where $\{s(t), x(t), y(t), a(t)\}$ is an arbitrary set of functions that could be time dependent. In this case, equation (14) gives $q = x + (1-a)Q$ and $p = y + aP$. Thus, function $S(Q, P, t)$ induces a set of well known quantization mappings, $\hat{q} \to \breve{Q}_+ = q + \breve{Q}$, and $\hat{p} \to \breve{P}_+ = p + \breve{P}$. Setting $s(t) = x(t) = y(t) = 0$, we obtain for $a = -1$, 0 and $\frac{1}{2}$ the Emch, van Hove and symmetric mappings, respectively [21, 22]. $x(t)$, $y(t)$, and $a(t)$ are free functions (or parameters) that induce different quantization schemes.

In addition, we have $\overline{q} = q$, $X = 0$, $\overline{p} = p$, and $Y = 0$. Since $X_{nn} = 1$ and $Y_{nn} = 1$ (see appendix B), equations (26) and (28) reduce to

$$\overset{2k+1}{\mathcal{F}_\mu}(Q, P, t) = \overset{2k+1}{F_\mu}(q, t), \overset{2k}{\mathcal{F}_\nu}(Q, P, t) = \overset{2k}{F_\nu}(p, t). \tag{38a}$$

Similarly, equations (34) and (35) become

$$\overset{2,3}{\mathcal{F}}_{\nu,\mu}(Q, P, t) = \frac{1}{\nu!\mu!}\sum_{n=0}^{\infty}(\lambda_{23})^n \frac{(n+\nu)!(n+\mu)!}{n!}\overset{2}{F}_{n+\nu}(p, t)\overset{3}{F}_{n+\mu}(q, t) \tag{38b}$$

$$\overset{5,4}{\mathcal{F}}_{\nu,\mu}(Q, P, t) = \frac{1}{\nu!\mu!}\sum_{n=0}^{\infty}(\lambda_{54})^n \frac{(n+\nu)!(n+\mu)!}{n!}\overset{4}{F}_{n+\nu}(p, t)\overset{5}{F}_{n+\mu}(q, t) \tag{38c}$$

where $\lambda_{23} := -\mathrm{i}\hbar(1-a)$, $\lambda_{54} := \mathrm{i}\hbar a$, and the coefficients F_ν are given by (17*b*).

Using (38*a*)–(38*c*) in (36), we obtain

$$\underset{+}{\mathcal{H}}\left(\breve{Q}_+, \breve{P}_+, t\right) = \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{1}{\nu!\mu!}\left(\left(\frac{1}{a}\frac{\partial}{\partial P}\right)^\nu \left(\frac{1}{1-a}\frac{\partial}{\partial Q}\right)^\mu K(Q, P, t)\right)\breve{P}^\nu\breve{Q}^\mu \tag{39}$$

where the generalized Hamiltonian is given by

$$K(Q, P, t) = H_+(x + (1-a)Q, y + aP, t) \tag{40a}$$

with

$$H_+(q, p, t) := \overset{0}{F}(p, t) + \overset{1}{F}(q, t) + \sum_{r=0}^{\infty}(-\mathrm{i}\hbar(1-a))^r r!\overset{2}{F}_r(p, t)\overset{3}{F}_r(q, t)$$
$$+\sum_{r=0}^{\infty}(\mathrm{i}\hbar a)^r r!\overset{5}{F}_r(q, t)\overset{4}{F}_r(p, t). \tag{40b}$$

Observe that $K(Q, P, t)$ depends on $\hbar$, except for the standard Hamiltonian, $\overset{0}{F}(p, t) + \overset{1}{F}(q, t)$.

5. Phase-space form of the Schrödinger equation

Let us apply the herein proposed 'classicalization' method to the Schrödinger equation. For a system with Hamiltonian $H(\hat{q}, \hat{p}, t)$ and initial state $|\psi(t_0)\rangle$, this equation describes the time evolution of quantum-mechanical state $|\psi(t)\rangle$. Among the infinite number of ways in which the quantum dynamics can be formulated (Heisenberg, interaction, etc), we define a *phase-space picture* by the transformation

$$|\psi_+(Q, P, t)\rangle := \exp\left(-\frac{\mathrm{i}}{\hbar}S(Q, P, t)\right)\underset{+}{\hat{D}}(Q, P)\,|\psi(t)\rangle \tag{41}$$

where $S(Q, P, t)$ is an *arbitrary* smooth phase-space function. We assume that free parameters Q and P are either time dependent (subsection 5.1) or time independent (subsection 5.2).

5.1. Time-dependent parameters $Q(t)$ and $P(t)$

Let Q and P be differentiable functions of time t, i.e. $Q(t)$ and $P(t)$. From now on we denote the position and momentum initial values as $Q_0 := Q(t_0)$ and $P_0 := P(t_0)$. To find the equation of motion for ket $|\psi_+(Q, P, t)\rangle$, we use the relation

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} \underset{+}{\hat{D}}(Q, P) = \underset{+}{\hat{D}}(Q, P) \left[-Q(\mathrm{d}P/\mathrm{d}t) + (\mathrm{d}P/\mathrm{d}t)\hat{q} - (\mathrm{d}Q/\mathrm{d}t)\hat{p}\right]. \tag{42}$$

Applying equations (15) and (16), we can write the Schrödinger equation as

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi_+(Q, P, t)\rangle = \left[\frac{\mathrm{d}S}{\mathrm{d}t} + \frac{\mathrm{d}P}{\mathrm{d}t}\left(\breve{Q}_+ - Q\right) - \frac{\mathrm{d}Q}{\mathrm{d}t}\breve{P}_+ + \underset{+}{\mathcal{H}}\left(\breve{Q}_+, \breve{P}_+, t\right)\right] |\psi_+(Q, P, t)\rangle \tag{43}$$

where $\breve{Q}_+ = q + \breve{Q}$, $\breve{P}_+ = p + \breve{P}$, and q and p are given by (14).

Until now all of the above expressions are valid for every set of *free parameters* $\{S, Q, P\}$. Note that the right-hand side of (43) is an expansion of the form $\alpha\breve{1} + \beta\breve{Q} + \gamma\breve{P} + \cdots$. Thus, we can simplify (43) by choosing $\{S, Q, P\}$ so that coefficients α, β and γ vanish for all time t. In this way, we obtain *generalized Hamilton equations*,

$$\mathrm{d}Q/\mathrm{d}t = K_{10}(Q, P, t) \qquad \mathrm{d}P/\mathrm{d}t = -K_{01}(Q, P, t) \tag{44}$$

and $S(Q, P, t)$ is determined by solving the time-dependent quantum *Hamilton–Jacobi equation*

$$\partial S/\partial t + K(Q, P, t) = 0 \tag{45}$$

where $\partial/\partial t := (\partial/\partial t)_{(Q,P)}$ is the time rate of change at a fixed phase-space point (Q, P).

We return to the Schrödinger equation (43), which reduces to

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi_+(Q, P, t)\rangle = \underset{+}{\mathcal{D}}\left(Q, P, \breve{Q}, \breve{P}, t\right) |\psi_+(Q, P, t)\rangle \tag{46a}$$

with initial state at t_0 given by

$$|\psi_+(Q_0, P_0, t_0)\rangle := \exp\left[-\frac{\mathrm{i}}{\hbar} S(Q_0, P_0, t_0)\right] \underset{+}{\hat{D}}(Q_0, P_0) |\psi(t_0)\rangle. \tag{46b}$$

Here, the system's quantum dynamics along the phase-space trajectory $(Q(t), P(t))$ is determined by

$$\underset{+}{\mathcal{D}}\left(Q, P, \breve{Q}, \breve{P}, t\right) := \underset{+}{\mathcal{H}}\left(\breve{Q}_+, \breve{P}_+, t\right) - K(Q, P, t) - \hbar L(Q, P, \breve{Q}, \breve{P}, t) \tag{47}$$

which can be expanded as (36), provided that contributions of multi-indices $(\nu, \mu) = (0, 0)$ and $(\nu, \mu) = (0, 1), (1, 0)$ are excluded. Here,

$$\hbar L(Q, P, \breve{Q}, \breve{P}, t) := K_{10}(Q, P, t)\breve{P} + K_{01}(Q, P, t)\breve{Q} \tag{48}$$

is the *generalized Liouville operator*. Note that as consequence of restriction $\nu + \mu \neq 0, 1$, the first term in (47) is of the order of $\hbar^2$. Furthermore, from (36) and (47) we conclude that for a standard Hamiltonian, $\underset{+}{\mathcal{D}}\left(Q, P, \breve{Q}, \breve{P}, t\right)$ assumes a simple form lacking mixed-derivative products of position and momentum, $\breve{P}^\nu \breve{Q}^\mu$.

5.2. Time-independent parameters Q and P

A second alternative to deal with (43) is to assume that phase-space parameters Q and P are time-independent, i.e. $\mathrm{d}Q/\mathrm{d}t = \mathrm{d}P/\mathrm{d}t = 0$. In this case, starting from (43) we obtain (45), but instead of (46*a*), we get the following motion equation

$$\mathrm{i}\hbar(\partial/\partial t)\,|\psi_+(Q,P,t)\rangle = \big[\hbar L\big(Q,P,\check{Q},\check{P},t\big) + \underset{+}{\mathcal{D}}\big(Q,P,\check{Q},\check{P},t\big)\big]\,|\psi_+(Q,P,t)\rangle \qquad (49a)$$

with initial condition, see (41),

$$|\psi_+(Q,P,t_0)\rangle := \exp\left(-\frac{\mathrm{i}}{\hbar}S(Q,P,t_0)\right)\underset{+}{\hat{D}}(Q,P)\,|\psi(t_0)\rangle. \qquad (49b)$$

Note that although we are working in this subsection with time-independent parameters Q and P, the generalized Liouville operator (48) can be written as

$$\hbar L(Q,P,\check{Q},\check{P},t) := (\mathrm{d}Q/\mathrm{d}t)\check{P} - (\mathrm{d}P/\mathrm{d}t)\check{Q} \qquad (50)$$

where $\mathrm{d}Q(t)/\mathrm{d}t$ and $\mathrm{d}P(t)/\mathrm{d}t$ are given by the generalized Hamilton equations (44). As $\partial/\partial t$ denotes the time rate of change at a fixed phase-space point, $(\mathrm{d}/\mathrm{d}t)$ is the time rate of change seen by an observer *moving* with $(Q(t), P(t))$, along the phase-space trajectory. The chain rule yields the following relationship between them:

$$\mathrm{i}\hbar\mathrm{d}/\mathrm{d}t = \mathrm{i}\hbar\partial/\partial t - \hbar L(Q,P,\check{Q},\check{P},t). \qquad (51)$$

Thus, quantum dynamics in phase space can be described by (46*a*) and (46*b*) or by (49*a*) and (49*b*). These are exact equations that closely resemble classical statistical mechanics. However, instead of involving probability distributions, they use kets $|\psi_+(Q,P,t)\rangle$, parametrized by (Q, P) phase-space points.

5.3. Classical limit

Let us comment about the classical limit of this treatment. Although the following argument has general validity, we use a specific example for comprehension purposes. Let us consider the Hamilton operator (31) and the symbol $H_+(q,p,t)$ given by (40*b*), which is a power series in $\hbar$. Since (47) and (48) relate the phase-space operator $\underset{+}{\mathcal{D}}$ with the symbol $H_+(q,p,t)$, the first term in $\underset{+}{\mathcal{D}}$ is due to multi-indices $(\nu,\mu) = (2,0), (0,2), (1,1)$. They provide contributions of the order of $\hbar^2 \times H_+(q,p,t)$. Although $H_+(q,p,t)$ is $\hbar$ dependent, we obtain low-order contributions in $\hbar$ from (45*a*) by disregarding $\underset{+}{\mathcal{D}}$; that is,

$$\mathrm{i}\hbar(\partial/\partial t)\left|\psi_+^{cl}(Q,P,t)\right\rangle = \hbar L(Q,P,\check{Q},\check{P},t)\left|\psi_+^{cl}(Q,P,t)\right\rangle \qquad (52)$$

is the quantum-mechanical equation of motion reduced to order $\hbar^0$ and with initial state (49*b*) at time t_0. Along the phase-space trajectory, equation (52) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi_+^{cl}(Q,P,t)\right\rangle = 0 \qquad (53)$$

with initial state (46*b*) at time t_0. We obtain the solutions of the *generalized Liouville equation* (52) by solving the underlying ordinary differential equations (44), because it is a partial differential equation of first order in the variables (Q,P,t). Note that, in the particular case of a standard Hamiltonian, $\overset{0}{T}(p,t) + \overset{1}{V}(q,t)$, the symbol $H_+(q,p,t)$ and the generalized Liouville operator $L(Q,P,\check{Q},\check{P},t)$ are $\hbar$ independent. In the general case (40*b*), the symbol

$H_+(q, p, t)$ is $\hbar$ dependent, but if a further approximation is desired we can let $\hbar \to 0$ in (40*b*) to obtain an $\hbar$-independent symbol.

Equations (52) and (53) describe low-order dynamics in $\hbar$ (namely, $\hbar^0$). They are the quantum analogue of the well known classical Liouville equation [23]. Thus, in equations (46*a*) and (49*a*), the operator $\underset{+}{\mathcal{D}}\left(Q, P, \breve{Q}, \breve{P}, t\right)$ is responsible for the quantum effects of order $\hbar^n$, $n \geqslant 1$.

6. Coherent-state representation

6.1. definition

Now we consider phase-space wavefunctions defined by Glauber coherent states [24, 25] and denoted here as $|Z(Q^*, P^*)\rangle$. $Z(Q^*, \pm P^*) := \kappa_0 Q^* \pm \mathrm{i}\chi_0 P^*$ defines an arbitrary complex number, where $\kappa_0 := (\sqrt{2}q_0)^{-1}$, $\chi_0 := (\sqrt{2}p_0)^{-1}$, q_0 and p_0 are position and momentum units restricted by the condition $q_0 p_0 = \hbar$. Parameters $Q = Q_R + \mathrm{i}Q_I$ and $P = P_R + \mathrm{i}P_I$ may be complex. From here on we use auxiliary functions

$$M(Q) := \left(\sqrt{\pi}q_0\right)^{-f/2} \exp\left[-\left(\kappa_0 Q\right)^2\right] \tag{54a}$$

$$\tilde{M}(P) := \left(\sqrt{\pi}p_0\right)^{-f/2} \exp\left[-\left(\chi_0 P\right)^2\right] \tag{54b}$$

with $Q^2 := Q_1^2 + Q_2^2 + \cdots + Q_f^2$ and $P^2 := P_1^2 + P_2^2 + \cdots + P_f^2$.

We represent the normalized ground state of the f-dimensional harmonic oscillator, also called the vacuum state, by $|0\rangle := |Z = 0\rangle = |n = 0\rangle$. Thus, we define state $|Z(Q^*, P^*)\rangle$ by the application of Weyl operator to this vacuum state, obtaining

$$\begin{aligned} \left|Z(Q^*, P^*)\right\rangle &:= \hat{D}(Q^*, P^*)|0\rangle \\ &= \left[(\pi\hbar)^{f/2} M\left(Q^*\right) \tilde{M}\left(P^*\right)\right]^{1/2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left(\kappa_0 Q^* + \mathrm{i}\chi_0 P^*\right)^n |n\rangle \end{aligned} \tag{55a}$$

where we have used the relations $\hat{q} = q_0(\hat{a}^+ + \hat{a})/\sqrt{2}$, $\hat{p} = \mathrm{i}p_0(\hat{a}^+ - \hat{a})/\sqrt{2}$ and properties of annihilation and creation operators, $\hat{a}$ and $\hat{a}^+$. Adjoint equation of (55*a*) is given by

$$\begin{aligned} \left\langle Z(Q^*, P^*)\right| &= \langle 0|\, \hat{D}(-Q, -P) \\ &= \left[(\pi\hbar)^{f/2} M(Q)\tilde{M}(P)\right]^{1/2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left(\kappa_0 Q - \mathrm{i}\chi_0 P\right)^n \langle n|. \end{aligned} \tag{55b}$$

In accordance with equations (1*a*) and (1*b*) and (15), let us introduce kets and bras $|Q, P\rangle_\pm$ and $_\pm\langle Q, P|$, which differ from $|Z(Q^*, P^*)\rangle$ and $\langle Z(Q^*, P^*)|$ by phase factors:

$$\begin{aligned} {}_\pm\langle Q, P| &:= \exp\left(\mp\frac{\mathrm{i}}{\hbar} S(Q, P, t)\right) \langle 0|\, \underset{\pm}{\hat{D}}(Q, P) \\ &= \exp\left(\mp\frac{\mathrm{i}}{\hbar} S(Q, P, t)\right) \underset{\pm}{w}\left(Q, \frac{1}{2}P\right) \left\langle Z(Q^*, P^*)\right| \end{aligned} \tag{56a}$$

$$\begin{aligned} |Q, P\rangle_\pm &:= \exp\left(\pm\frac{\mathrm{i}}{\hbar} S^*(Q, P, t)\right) \underset{\mp}{\hat{D}}(-Q^*, -P^*)|0\rangle \\ &= \exp\left(\pm\frac{\mathrm{i}}{\hbar} S^*(Q, P, t)\right) \underset{\mp}{w}\left(Q^*, \frac{1}{2}P^*\right) \left|Z(Q^*, P^*)\right\rangle. \end{aligned} \tag{56b}$$

Thus, an arbitrary quantum-mechanical state $|\psi(t)\rangle$ admits coherent-state representations defined by scalar products

$$\psi_{\pm}(Q,P,t) := {}_{\pm}\langle Q,P|\psi(t)\rangle = \exp\left[\mp\frac{\mathrm{i}}{\hbar}S(Q,P,t)\right]\langle 0|\,\underset{\pm}{\hat{D}}(Q,P)\,|\psi(t)\rangle \tag{57a}$$

$$= \exp\left[\mp\frac{\mathrm{i}}{\hbar}S(Q,P,t)\right]\underset{\pm}{w}\left(Q,\frac{1}{2}P\right)\left[(\pi\hbar)^{f/2}M(Q)\tilde{M}(P)\right]^{1/2} \times\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}(\kappa_0 Q-\mathrm{i}\chi_0 P)^n\langle n|\psi(t)\rangle. \tag{57b}$$

Here $\langle n|\psi(t)\rangle$ is the scalar product of the state $|\psi(t)\rangle$ and eigenstate $|n\rangle$ of the f-dimensional harmonic oscillator. Results in (57*a*) and (57*b*) are standard [24–26], except for the presence of phase-space factors $\exp[\mp\mathrm{i}S(Q,P,t)/\hbar]\,\underset{\pm}{w}(Q,P/2)$. These are required to correctly represent in phase space the eigenvalue and Schrödinger equations, when applying the herein presented 'classicalization' procedure. Here, we point out that (57*a*) and (1*b*) lead to the relation

$$\exp\left(-\frac{\mathrm{i}}{\hbar}S(Q,P,t)\right)\psi_{-}(Q,P,t) = \underset{-}{w}(Q,P)\exp\left(\frac{\mathrm{i}}{\hbar}S(Q,P,t)\right)\psi_{+}(Q,P,t). \tag{58}$$

The expectation value of the density operator $\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|$ in the coherent state $|Z(Q^*,P^*)\rangle$ gives the *Husimi function*, $\rho(Q,P,t) := \langle Z(Q^*,P^*)|\,\hat{\rho}(t)\,|Z(Q^*,P^*)\rangle$. Then, we can use (56*a*), (56*b*) and (57*a*), to write the Husimi function as $\rho(Q,P,t) = \exp(\pm\Delta(Q,P,t))\,|\psi_{\pm}(Q,P,t)|^2$, with

$$\Delta(Q,P,t) := (\mathrm{i}/\hbar)\left[S(Q,P,t)-QP/2-(S(Q,P,t)-QP/2)^*\right].$$

The factor $\exp(\pm\Delta(Q,P,t))$ takes the value one except in the case of complex functions S, Q or P.

6.2. Application to the Schrödinger equation

Multiplying both sides of (36) with $\langle 0|$, we obtain

$$\psi_{+}(Q,P,t) := \langle 0|\psi_{+}(Q,P,t)\rangle = \exp[-(\mathrm{i}/\hbar)S(Q,P,t)]\,\langle 0|\,\underset{+}{\hat{D}}(Q,P)\,|\psi(t)\rangle. \tag{59}$$

Similarly, motion equations (46*a*) and (49*a*) can be written in coherent-state representations by substituting: $|\psi_{+}(Q,P,t)\rangle \to \psi_{+}(Q,P,t)$. Thus, equation (49*a*) becomes

$$\mathrm{i}\hbar(\partial/\partial t)\psi_{+}(Q,P,t) = \left[\hbar L\big(Q,P,\check{Q},\check{P},t\big)+\underset{+}{\mathcal{D}}\big(Q,P,\check{Q},\check{P},t\big)\right]\psi_{+}(Q,P,t) \tag{60}$$

with initial state $\psi_{+}(Q,P,t_0)$ at time t_0.

6.3. Evaluation of matrix elements

Now we are interested in evaluating $\langle\varphi(t)|\,\hat{1}\,\underset{+}{F}(\hat{q},\hat{p},t)\,|\psi(t)\rangle$ for arbitrary operators of the form (8*a*) or (9). The unity operator $\hat{1}$ admits a resolution in terms of coherent states $|Q,P\rangle_{+}$:

$$\hat{1} = (2\pi\hbar)^{-f}\int \mathrm{d}Q\,\mathrm{d}P\,|Q,P\rangle_{+}\exp\left[-\frac{\mathrm{i}}{\hbar}S(Q,P,t)\right]\langle 0|\,\underset{+}{\hat{D}}(Q,P). \tag{61a}$$

So, using (61*a*), (10*a*), (15) and (57*a*), we obtain

$$\langle\varphi(t)|\,\underset{+}{F}(\hat{q},\hat{p},t)\,|\psi(t)\rangle = (2\pi\hbar)^{-f}\int \mathrm{d}Q\,\mathrm{d}P\,\varphi_{+}{}^{*}(Q,P,t)\,\underset{+}{\mathcal{F}}\big(\check{Q}_{+},\check{P}_{+},t\big)\,\psi_{+}(Q,P,t). \tag{61b}$$

6.4. Application to the eigenvalue equation

Let us consider the eigenvalue equation of an arbitrary operator $\underset{+}{F}(\hat{q}, \hat{p}, t)$. Applying $\exp[-\mathrm{i}S(Q, P, t)/\hbar]\,\langle 0|\,\underset{+}{\hat{D}}(Q, P)$ to both sides of $\underset{+}{F}(\hat{q}, \hat{p}, t)\,|\psi\rangle = E\,|\psi\rangle$, and using (10*a*) and (15), we convert the eigenvalue equation to

$$\underset{+}{\mathcal{F}}\left(\breve{Q}_+, \breve{P}_+\right)\psi_+(Q, P) = E\psi_+(Q, P). \tag{62}$$

Consider, for instance, eigenvalue equation

$$\left(\kappa_0\hat{q} + \mathrm{i}\chi_0\hat{p}\right)\left|Z(Q'', P'')\right\rangle = Z(Q'', P'')\left|Z(Q'', P'')\right\rangle.$$

It can be written as

$$\left(\kappa_0\breve{Q}_+ + \mathrm{i}\chi_0\breve{P}_+\right)\psi_+(Q, P; Q'', P'') = Z(Q'', P'')\psi_+(Q, P; Q'', P'')$$

with

$$\psi_+(Q, P; Q'', P'') = \exp[-\mathrm{i}S(Q, P, t)/\hbar]\,\langle 0|\,\underset{+}{\hat{D}}(Q, P)\left|Z(Q'', P'')\right\rangle$$

$\breve{P}_+ = \partial S/\partial Q + \breve{P}$, and $\breve{Q}_+ = Q - \partial S/\partial P + \breve{Q}$. Note that relations (55*a*) and

$$\hat{D}\left(Q', P'\right)\hat{D}\left(Q'', P''\right) = \exp\left[-\frac{\mathrm{i}}{2\hbar}\left(Q'P'' - P'Q''\right)\right]\hat{D}\left(Q' + Q'',\; P' + P''\right) \tag{63}$$

are useful to express $\psi_+(Q, P; Q'', P'')$ as a scalar product of two Glauber's coherent states.

6.5. Wavefunctions in position and momentum space

In this subsection we consider the relation between the coherent-state representation and wavefunctions in position and momentum space. Since the unity operator $\hat{1}$ admits a resolution in terms of position eigenstates $|q\rangle$, with $\hat{q}\,|q\rangle = q\,|q\rangle$, and $\langle q + Q| = \langle q|\exp\left((\mathrm{i}/\hbar)Q\hat{p}\right)$, this completeness relation allows one to write (57*a*) as

$$\begin{aligned}\psi_+(Q, P, t) &= \exp\left(-\frac{\mathrm{i}}{\hbar}S(Q, P, t)\right)\langle 0|\,\underset{+}{\hat{D}}(Q, P)\,|\psi(t)\rangle \\ &= \exp\left(-\frac{\mathrm{i}}{\hbar}S(Q, P, t)\right)\int \mathrm{d}q\;\underset{-}{w}(q, P)\,M(q)\psi(q + Q, t) \qquad (64)\end{aligned}$$

where $\psi(q, t) := \langle q|\psi(t)\rangle$ is the position-space wavefunction and $\langle q\,|Z = 0\rangle = M(q)$. By using the Taylor series expansion of $\psi(q + Q, t)$, we obtain

$$\psi_+(Q, P, t) = \exp\left(-\frac{\mathrm{i}}{\hbar}S(Q, P, t)\right)(2\pi\hbar)^{f/2}\sum_{m=0}^{\infty}\frac{1}{m!}\psi^{(m)}(Q, t)\tilde{J}_m(P) \tag{65}$$

with $\psi^{(m)}(Q, t) := (\partial/\partial Q)^m\,\psi(Q, t)$, and

$$\begin{aligned}\tilde{J}_m(P) &:= (2\pi\hbar)^{-f/2}\int \mathrm{d}q\;\underset{-}{w}(q, P)\,M(q)q^m \\ &= (\mathrm{i}\hbar\partial/\partial P)^m\,\tilde{J}_0(P) = \left(-\mathrm{i}q_0/\sqrt{2}\right)^m H_m(\chi_0 P)\,\tilde{M}(P) \qquad (66)\end{aligned}$$

where $H_m(y)$ are Hermite polynomials and $\tilde{M}(p) = \langle p|Z = 0\rangle$ is given by (54*b*). As consequence of (65) and orthonormality of Hermite polynomials, we find the derivatives of position-space wavefunction ($\eta = 0, 1, 2, \ldots$)

$$\begin{aligned}\psi^{(\eta)}(Q, t) &= \left(2\sqrt{\pi}\,p_0\right)^{-f/2}\left(-\mathrm{i}\sqrt{2}\,q_0\right)^{-\eta} \\ &\quad\times(2\pi\hbar)^{-f/2}\int \mathrm{d}P\,\exp\left(\frac{\mathrm{i}}{\hbar}S(Q, P, t)\right)\psi_+(Q, P, t)H_\eta(\chi_0 P). \qquad (67)\end{aligned}$$

Similarly, by using the eigenkets of momentum operator $\hat{p}$, $\hat{p}\,|p\rangle = p\,|p\rangle$, we obtain from the second equation (57*a*) the expansion

$$\psi_-(Q, P, t) = \exp\left(\frac{\mathrm{i}}{\hbar}S(Q, P, t)\right)(2\pi\hbar)^{f/2}\sum_{m=0}^{\infty}\frac{1}{m!}\tilde{\psi}^{(m)}(P, t)J_m(Q) \tag{68}$$

with $\tilde{\psi}(P) := \langle p\,|\psi(t)\rangle$, and

$$J_m(Q) = \left(\mathrm{i}p_0/\sqrt{2}\right)^m H_m(\kappa_0 Q)\,M(Q). \tag{69}$$

The derivatives of momentum-space wavefunction are given by ($\eta = 0, 1, 2, \ldots$)

$$\tilde{\psi}^{(\eta)}(P, t) = \left(2\sqrt{\pi}\,q_0\right)^{-f/2}\left(\mathrm{i}\sqrt{2}\,p_0\right)^{-\eta} \times (2\pi\hbar)^{-f/2}\int \mathrm{d}Q\,\exp\left(-\frac{\mathrm{i}}{\hbar}S(Q, P, t)\right)\psi_-(Q, P, t)H_\eta(\kappa_0 Q). \tag{70}$$

Note that we can determine matrix elements by using the position completeness relation to write (see (61*b*))

$$\langle\varphi(t)|\,\underset{+}{F}(\hat{q}, \hat{p}, t)\,|\psi(t)\rangle = \int \mathrm{d}Q\,\varphi^*(Q, t)\,\underset{+}{F}\left(Q, \check{P}, t\right)\psi(Q, t) \tag{71}$$

and by employing (67) to evaluate the Q-derivatives of wavefunction $\psi(Q, t)$. A similar relation is obtained by using momentum completeness relation and (70).

6.6. Squeezed representation

At this point some comments concerning the relation of present method with Gaussian semiquantal dynamics are in order. Let us introduce the squeeze operator

$$\hat{\Gamma}(\beta) = \Gamma\left(\hat{q}, \hat{p}, \beta\right) := \exp\left[\frac{1}{2}\left(\beta\,\hat{a}^{+2} - \beta^*\hat{a}^2\right)\right] \tag{72a}$$

$$\hat{\Gamma}^+(\beta) = \hat{\Gamma}^{-1}(\beta) = \hat{\Gamma}\,(-\beta) \tag{72b}$$

where, for one degree of freedom, the squeeze parameter β is a complex number. The squeezed state is defined as [13]: $|Z(Q^*, P^*), \beta\rangle := \hat{D}(Q^*, P^*)\hat{\Gamma}(\beta)|0\rangle$.

For a system with f degree of freedom, we could consider instead of (41) the squeezed transformation

$$|\psi_\Gamma(Q, P, t)\rangle := \exp\left(-\frac{\mathrm{i}}{\hbar}S(Q, P, t)\right)\underset{+}{\hat{D}}(Q, P)\,\Gamma\left(\hat{q}, \hat{p}, \beta\right)|\psi(t)\rangle \tag{73}$$

where $\Gamma\left(\hat{q}, \hat{p}, \beta\right)$ denotes a suitable product of squeeze operators, and $\beta = (\beta_1, \beta_2, \ldots, \beta_f)$ is a set of f time-dependent parameters. By following a similar procedure to that of subsection 5.1 we could fix the set of free parameters $\{S, Q, P, \beta\}$ and obtain equations of motion for parameters β, besides equations (44) and (45). This result indicates that the present method is related to the squeezed state approach (semiquantum dynamics). This theory leads to a set of Hamilton equations describing expectations values and quantum fluctuations in an extended phase space [14–17].

In analogy with (59), multiplying both sides of (73) with $\langle 0|$, we obtain

$$\begin{aligned}\psi_\Gamma(Q, P, t) := \langle 0|\psi_\Gamma(Q, P, t)\rangle &= \exp\left(-\frac{\mathrm{i}}{\hbar}S(Q, P, t)\right)\langle 0|\,\underset{+}{\hat{D}}(Q, P)\Gamma\left(\hat{q}, \hat{p}, \beta\right)|\psi(t)\rangle \\ &= \exp\left(-\frac{\mathrm{i}}{\hbar}S(Q, P, t)\right)\underset{+}{\Gamma}(Q, P, \check{Q}, \check{P}, \beta)\,\langle 0|\,\underset{+}{\hat{D}}(Q, P)\,|\psi(t)\rangle\end{aligned}$$

where we use the intertwining relation (10*a*) to change from Hilbert-space operator $\Gamma\left(\hat{q}, \hat{p}, \beta\right)$ to phase-space operator $\underset{+}{\Gamma}(Q, P, \check{Q}, \check{P}, \beta)$. Finally, we get from (57*a*) and (15) that wavefunction $\psi_\Gamma(Q, P, t)$ is related to wavefunction (55) by transformation

$$\psi_\Gamma(Q, P, t) = \underset{+}{\Gamma}(\check{Q}_+, \check{P}_+, \beta)\psi_+(Q, P, t) \tag{74}$$

with $\check{Q}_+$ and $\check{P}_+$ given by (13*a*) and (13*b*).

7. Examples

7.1. Dynamics of a free particle

Let us consider in full detail the case of a free particle of mass m and Hilbert-space Hamilton operator $H(\hat{q}, \hat{p}, t) = \hat{p}^2/(2m)$. According to (16) or (31), the phase-space operator associated with $H(\hat{q}, \hat{p}, t)$ is $\underset{+}{\mathcal{H}}\left(\check{Q}_+, \check{P}_+, t\right) = \check{P}_+{}^2/(2m)$, where $\check{P}_+$ is giving by (13*b*) and (14). In this step let us assume that $S(Q, P, t)$ is an *arbitrary* smooth function.

Now, by using $\check{P}_+ = p(Q, P, t) + \check{P}$, $\left[Q, \check{P}\right] = \mathrm{i}\hbar\check{1}$, and $\check{P}\,(\partial S/\partial Q) = (\partial S/\partial Q)\,\check{P} - \mathrm{i}\hbar\left(\partial^2 S/\partial Q^2\right)$, we write $\underset{+}{\mathcal{H}}\left(\check{Q}_+, \check{P}_+, t\right)$ in the form (37):

$$\underset{+}{\mathcal{H}}\left(\check{Q}_+, \check{P}_+, t\right) = \frac{1}{2m}\left[(\partial S/\partial Q)^2 - \mathrm{i}\hbar\partial^2 S/\partial Q^2\right] + \frac{1}{m}\,(\partial S/\partial Q)\,\check{P} + \frac{1}{2m}\check{P}^2. \tag{75}$$

By comparing this expression with (37) we identify $K(Q, P, t)$, $K_{10}(Q, P, t)$, $K_{01}(Q, P, t)$ and $K_{20}(Q, P, t)$.

Now, we recall that an acceptable physical function $S(Q, P, t)$ must satisfy equation (45). Substituting the ansatz $S(Q, P, t) = -s(t) + y(t)Q$ into (45), we find that at every phase-space point (Q, P) the formula (45) is satisfied, if $y(t)$ is a time-independent constant (i.e. $y(t) = P_0$). That is, $S(Q, P, t) = -s(t) + P_0 Q$, with $s(t) = (t - t_0)P_0^2/(2m)$. The dynamical equations (44) become the standard Hamilton equation and, therefore, a phase-space trajectory is described by $Q = Q_0 + (P_0/m)(t - t_0)$ and $P = P_0$.

We are now interested in the quantum dynamics of the free particle as seen by an observer at a fixed phase-space point (Q, P). We can use either the formal equation (49*a*) and (49*b*) or the coherent state representation (60), with initial state $\psi_+(Q, P, t_0)$ at time t_0 (see (49*b*)). Equation (60) becomes

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_+(Q, P, t) = \left[\frac{P_0}{m}\check{P} + \frac{1}{2m}\check{P}^2\right]\psi_+(Q, P, t) \tag{76}$$

and one can immediately write the solution as

$$\psi_+(Q, P, t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\left(\frac{P_0}{m}\check{P} + \frac{1}{2m}\check{P}^2\right)(t - t_0)\right]\exp\left(-\frac{\mathrm{i}}{\hbar}P_0 Q\right)\langle 0|\,\underset{+}{\hat{D}}(Q, P)\,|\psi(t_0)\rangle \tag{77}$$

where we have used $S(Q, P, t_0) = P_0 Q$. We can see from (64) that $\langle 0|\,\underset{+}{\hat{D}}(Q', P)\,|\psi(t_0)\rangle = \exp(\mathrm{i}S(Q', P, t_0)/\hbar)\psi_+(Q', P, t_0)$, where $\psi_+(Q, P, t_0)$ is the initial phase-space state.

The phase-space wavefunction (77) can be calculated by recalling the identities [27, exercise 7.33]

$$F(Q) = \int_{-\infty}^{\infty} F(Q')\delta(Q' - Q)\,\mathrm{d}Q' \tag{78a}$$

$$\delta(Q' - Q) = (2\sqrt{\pi\sigma})^{-1}\exp\left(-\sigma\partial^2/\partial Q^2\right)\exp\left[-(Q' - Q)^2/(4\sigma)\right] \tag{78b}$$

where σ is a parameter and $F(Q)$ a function. Thus, we obtain

$$\exp\left(\sigma\partial^2/\partial Q^2\right) F(Q) = (2\sqrt{\pi\sigma})^{-1}\int_{-\infty}^{\infty}\exp\left[-(Q'-Q)^2/(4\sigma)\right]F(Q')\,\mathrm{d}Q'. \tag{78c}$$

For the calculation of (77), we also need to take advantage of the identity $F(Q-\zeta) = \exp(-\zeta\,\partial/\partial Q)\,F(Q)$, where ζ is constant with respect to Q. Finally, writing $\zeta(t) = (P_0/m)\,(t-t_0)$ and $q_0^2\sigma(t) = \mathrm{i}\hbar\,(t-t_0)\,/(2m)$, equation (77) assumes the form

$$\psi_+(Q,P,t) = \frac{1}{2\sqrt{\pi q_0^2\sigma(t)}} \times\int_{-\infty}^{\infty}\exp\left[-\frac{\left(Q'-Q_0\right)^2}{4q_0^2\sigma(t)}\right]\exp\left(-\frac{\mathrm{i}}{\hbar}P_0Q'\right)\langle 0|\,\underset{+}{\hat{D}}(Q',P_0)\,|\psi(t_0)\rangle\,\mathrm{d}Q' \tag{79}$$

with $Q_0 = Q-(t-t_0)P/m$, and $P_0 = P$. Using (64), it holds that $\langle 0|\,\underset{+}{\hat{D}}(Q',P)\,|\psi(t_0)\rangle = \exp(\mathrm{i}S(Q',P,t_0)/\hbar)\psi_+(Q',P,t_0)$, with $S(Q',P,t_0) = P_0Q'$, and $\psi_+(Q',P,t_0)$ the initial phase-space state. Making use of delta-function representation,

$$\delta(x-y) = \lim_{a\to\infty}\sqrt{\frac{a}{\pi\hbar\mathrm{i}}}\exp\left(\frac{\mathrm{i}a}{\hbar}(x-y)^2\right) \tag{80}$$

in the limit $t\to t_0$, equation (79) reduces to initial state $\psi_+(Q,P,t_0)$.

Until now we have considered the quantum dynamics as seen by an observer at a fixed phase-space point (Q,P). The quantum dynamics as seen by an observer moving with $(Q(t),P(t))$, along the phase-space trajectory, is determined by solving the equation of motion (46*a*) and (46*b*). The solution of (46*a*) and (46*b*) is given by (79) provided we follow the phase-space trajectory: $Q\to Q = Q_0+\zeta(t)$ and $P\to P = P_0$. After this substitution, the right-hand side of (79) becomes a function of the initial conditions, (Q_0,P_0). However, due to the inverse relation $(Q_0,P_0) = (Q-\zeta(t),P)$, the observer moving with $(Q(t),P(t))$ recovers (79) and sees the phase-space wavefunction as a function of $Q(t)$, $P(t)$, and t.

Using (79) and (67), with $\eta = 0$, we may derive the wavefunction in position-space. After using the identities,

$$\langle 0|\,\underset{+}{\hat{D}}(Q',P_0)\,|\psi(t_0)\rangle = \int\mathrm{d}q\,\exp\left(-\frac{\mathrm{i}}{\hbar}P_0q\right)M(q)\psi\left(q+Q',t_0\right) \tag{81a}$$

$$\frac{1}{2\pi\hbar}\int\mathrm{d}P_0\exp\left(-\frac{\mathrm{i}}{\hbar}qP_0\right) = \delta(q) \tag{81b}$$

we obtain the equation

$$\psi\,(Q,t) = \int\mathrm{d}Q'\,K(Q,t|Q',t_0)\psi(Q',t_0). \tag{81c}$$

The propagator $K(Q,t|Q',t_0)$ is given by

$$K(Q,t|Q',t_0) := \frac{1}{2\sqrt{\pi q_0^2\sigma(t)}}\exp\left(-\frac{(Q'-Q)^2}{4q_0^2\sigma(t)}\right) \tag{81d}$$

and it is equivalent to the standard one [28, p 112].

In a similar manner, using (58), (70), (79), and the relation

$$\int\mathrm{d}Q'\exp\left(-\frac{\mathrm{i}}{\hbar}P_0Q'\right)\langle 0|\,\underset{+}{\hat{D}}(Q',P_0)\,|\psi(t_0)\rangle = 2\pi\hbar\left(\sqrt{\pi}\,p_0\right)^{-1/2}\tilde{\psi}(P_0,t_0) \tag{82a}$$

we obtain momentum-space wavefunction

$$\tilde{\psi}(P,t)=\exp\left(-\frac{\mathrm{i}}{\hbar}\frac{P^2}{2m}(t-t_0)\right)\tilde{\psi}(P,t_0). \qquad (82b)$$

7.2. Particle in a time-dependent linear potential

Consider the Hilbert-space Hamilton operator $H(\hat{q},\hat{p},t)=\hat{p}^2/(2m)-F(t)\hat{q}$. According to (16), or (31), and (13*a*) and (13*b*), we obtain

$$\underset{+}{\mathcal{H}}\left(\breve{Q}_+,\breve{P}_+,t\right)=\frac{1}{2m}\left[\left(\frac{\partial S}{\partial Q}\right)^2-\mathrm{i}\hbar\frac{\partial^2 S}{\partial Q^2}\right]-F(t)\left(Q-\frac{\partial S}{\partial P}\right)$$
$$+\frac{1}{m}\frac{\partial S}{\partial Q}\breve{P}-F(t)\breve{Q}+\frac{1}{2m}\breve{P}^2. \qquad (83)$$

Using the ansatz $S(Q,P,t)=-s(t)-x(t)P+y(t)Q$ one sees that equation (45) is satisfied at each phase-space point (Q,P), if $x(t)=x_0$. Generalized Hamilton equations (44) become $\mathrm{d}Q/\mathrm{d}t=(\partial S/\partial Q)/m=y(t)/m$, $\mathrm{d}P/\mathrm{d}t=F(t)$. We fix $y_0=y(t_0)$ as the initial impulse ($y_0=P_0$) in order to obtain Q and P as a Hamilton equations solution: $Q=Q_0+(P_0/m)(t-t_0)+u(t)/m$, $P=P_0+g(t)$, where $Q_0=Q(t_0)$ and $P_0=P(t_0)$ are the initial conditions. We choose constant x_0 as $x_0=0$ so that Q, P and $S(Q,P,t)$ reduce to the equations of a free particle, in the case $F(t)=0$ (example 7.1). All together, we have

$$y(t)=P_0+g(t) \qquad \mathrm{d}s(t)/\mathrm{d}t=y^2(t)/(2m) \qquad (84a)$$

$$s(t)=\frac{t-t_0}{2m}P_0^2+\frac{1}{m}u(t)P_0+\frac{1}{2m}v(t) \qquad (84b)$$

$$S(Q,P,t)=-s(t)+y(t)Q \qquad (84c)$$

$$g(t):=\int_{t_0}^{t}F(t')\,\mathrm{d}t' \qquad u(t):=\int_{t_0}^{t}g(t')\,\mathrm{d}t' \qquad (84d)$$

$$m\chi(t):=-\int_{t_0}^{t}\mathrm{d}t'(t'-t_0)F(t')=u(t)-(t-t_0)g(t) \qquad (84e)$$

$$v(t):=\int_{t_0}^{t}g^2(t')\,\mathrm{d}t'=g(t)u(t)-2m\gamma(t),\ 2m\gamma(t):=\int_{t_0}^{t}\mathrm{d}t'F(t')u(t'). \qquad (84f)$$

We now use equations (60) and (83) to describe the quantum dynamics by the equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_+(Q,P,t)=\left[\frac{y(t)}{m}\breve{P}-F(t)\breve{Q}+\frac{1}{2m}\breve{P}^2\right]\psi_+(Q,P,t). \qquad (85)$$

We can obtain the solution of (85) by applying the Magnus method after noting that $[\breve{Q},y(t)]=\mathrm{i}\hbar\breve{1}$. However, equation (85) can be easily solved by using the phase-space transformation $(Q,P)\leftrightarrow(Q_0,P_0)$ and the relations $\breve{P}=\breve{P}_0$, $\breve{Q}=\breve{Q}_0+(t-t_0)\breve{P}_0/m$, where $\breve{Q}_0=\mathrm{i}\hbar\partial/\partial P_0$ and $\breve{P}_0=-\mathrm{i}\hbar\partial/\partial Q_0$. Similarly, if we denote by $\partial/\partial_0 t:=(\partial/\partial t)_{(Q_0,P_0)}$ the time rate of change at a fixed phase-space point (Q_0,P_0), and use the chain rule, we obtain $\mathrm{i}\hbar\partial/\partial t=\mathrm{i}\hbar\partial/\partial_0 t+[P-(t-t_0)F(t)]\,\breve{P}_0/m-F(t)\breve{Q}_0$. In this way (85) simplifies drastically: $\mathrm{i}\hbar\partial\psi_+(Q,P,t)/\partial_0 t=\breve{P}_0^2/(2m)\psi_+(Q,P,t)$. The solution of this equation is given by

$$\psi_+(Q,P,t)=\exp\left(-\frac{\mathrm{i}}{\hbar}\frac{t-t_0}{2m}\breve{P}_0^2\right)\underset{-}{w}(Q_0,P_0)\,\langle 0|\,\underset{+}{\widehat{D}}(Q_0,P_0)\,|\psi(t_0)\rangle \qquad (86a)$$

$$=\frac{1}{2\sqrt{\pi q_0^2\sigma(t)}}\int_{-\infty}^{\infty}\mathrm{d}Q'\exp\left(-\frac{\left(Q'-Q_0\right)^2}{4q_0^2\sigma(t)}\right)\underset{-}{w}(Q',P_0)\,\langle 0|\,\underset{+}{\widehat{D}}\left(Q',P_0\right)|\psi(t_0)\rangle$$

$$(86b)$$

with $q_0^2\sigma(t) := \mathrm{i}\hbar\,(t-t_0)\,/(2m)$, $Q_0 = Q-(t-t_0)P/m-\chi(t)$ and $P_0 = P-g(t)$.

The propagator $K(Q,t\,|Q',t_0)$ and the momentum wavefunction become

$$K(Q,t\,|Q',t_0) = \frac{1}{2\sqrt{\pi q_0^2\sigma(t)}}\exp\left(\frac{\mathrm{i}}{\hbar}\left(g(t)Q-\frac{v(t)}{2m}\right)\right) \times \exp\left(-\frac{(Q-u(t)/m-Q')^2}{4q_0^2\sigma(t)}\right) \tag{87}$$

$$\tilde{\psi}(P,t) = \exp\left(\frac{\mathrm{i}}{\hbar}\left(\frac{1}{2}\chi(t)g(t)+\gamma(t)\right)\right)\exp\left(-\frac{\mathrm{i}}{\hbar}\left[\frac{t-t_0}{2m}P^2+\chi(t)P\right]\right)\tilde{\psi}\,(P-g(t),t_0). \tag{88}$$

For the simple case of a particle of mass m in a time-independent potential $-F\hat{q}$, where F is a constant, the propagator and momentum wavefunction reduce to standard results:

$$K(Q,t\,|Q',t_0) = \sqrt{\frac{m}{2\pi\hbar\mathrm{i}\tau}}\exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{m}{2\tau}(Q-Q')^2+\frac{F\tau}{2}(Q+Q')-\frac{F^2\tau^3}{24m}\right)\right] \tag{89}$$

$$\tilde{\psi}(P,t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\frac{P^3-(P-F\tau)^3}{6mF}\right]\tilde{\psi}\,(P-F\tau,t_0) \tag{90}$$

where $\tau = t-t_0$.

7.3. Harmonic oscillator

For this system $\hat{H} = H(\hat{q},\hat{p}) = \hat{p}^2/(2m)+(m\omega^2/2)\hat{q}^2$, and

$$\underset{+}{\mathcal{H}}\left(\breve{Q}_+,\breve{P}_+,t\right) = \frac{1}{2m}\left[\left(\frac{\partial S}{\partial Q}\right)^2-\mathrm{i}\hbar\frac{\partial^2 S}{\partial Q^2}\right]+\frac{1}{2}m\omega^2\left[\left(Q-\frac{\partial S}{\partial P}\right)^2-\mathrm{i}\hbar\frac{\partial^2 S}{\partial P^2}\right] +\frac{1}{m}\frac{\partial S}{\partial Q}\breve{P}+m\omega^2\left(Q-\frac{\partial S}{\partial P}\right)\breve{Q}+\frac{1}{2m}\breve{P}^2+\frac{1}{2}m\omega^2\breve{Q}^2. \tag{91}$$

Equation (45) for the function $S(Q,P,t)$ can be solved using the following ansatz:

$$S(Q,P,t) = -s(t)+y(t)Q-\frac{1}{2}m\Omega(t)Q^2. \tag{92}$$

In this way we have the equations $\mathrm{d}\Omega/\mathrm{d}t = \Omega^2+\omega^2$, $\mathrm{d}y/\mathrm{d}t = \Omega(t)y$, and $\mathrm{d}s/\mathrm{d}t = \left(y^2+\mathrm{i}\hbar m\Omega\right)/(2m)$. The dynamical frequency $\Omega(t)$ is given by

$$\Omega(t) = \omega\tan\left(\omega(t-t_0)+\sigma_0\right) \tag{93a}$$

where $\tau = t-t_0$, and the phase σ_0 fixes the initial value of $\Omega(t_0)$ as $\omega\tan\sigma_0$. By noting the relation

$$\int_{t_0}^{t}\mathrm{d}t'\,\Omega(t') = -\ln\left(\frac{\cos(w\tau+\sigma_0)}{\cos\sigma_0}\right)$$

it then follows that

$$y(t) = \frac{y_0\cos\sigma_0}{\cos(\omega\tau+\sigma_0)} \tag{93b}$$

$$s(t) = \frac{(y_0\cos\sigma_0)^2}{2mw}\left(\tan(\omega\tau+\sigma_0)-\tan\sigma_0\right)-\frac{1}{2}\mathrm{i}\hbar\ln\left(\frac{\cos(\omega\tau+\sigma_0)}{\cos\sigma_0}\right). \tag{93c}$$

The generalized Hamilton equations become $\mathrm{d}Q/\mathrm{d}t = y(t)/m - \Omega(t)Q$, $\mathrm{d}P/\mathrm{d}t = -m\omega^2 Q$, and their solutions are given by

$$Q(t) = \left(Q_0 - \frac{y_0\cos\sigma_0\sin\sigma_0}{mw}\right)\frac{\cos(\omega\tau+\sigma_0)}{\cos\sigma_0} + \frac{y_0\cos\sigma_0}{mw}\sin(\omega\tau+\sigma_0) \tag{94a}$$

$$P(t) = P_0 - mw\left(Q_0 - \frac{y_0\cos\sigma_0\sin\sigma_0}{mw}\right)\frac{\sin(w\tau+\sigma_0)-\sin\sigma_0}{\cos\sigma_0} + y_0\cos\sigma_0\left(\cos(w\tau+\sigma_0)-\cos\sigma_0\right) \tag{94b}$$

with $Q(t_0) = Q_0$ and $P(t_0) = P_0$. Functions $Q(t)$ and $P(t)$ depend on the parameters y_0 and σ_0, and, in general, they differ from the solution of Hamilton's canonical equations. However, they can also be cast in the form

$$Q(t) = Q_0\cos(w\tau) + \frac{1}{mw}\left(y_0\cos^2\sigma_0 - (mwQ_0 - y_0\cos\sigma_0\sin\sigma_0)\tan\sigma_0\right)\sin(w\tau) \tag{95a}$$

$$P(t) = P_0 - \left(y_0\cos^2\sigma_0 - (mwQ_0 - y_0\cos\sigma_0\sin\sigma_0)\tan\sigma_0\right) - mwQ_0\sin(w\tau) + \left(y_0\cos^2\sigma_0 - (mwQ_0 - y_0\cos\sigma_0\sin\sigma_0)\tan\sigma_0\right)\cos(w\tau). \tag{95b}$$

Thus, equations (95*a*) and (95*b*) reduce to the *standard form* describing the classical dynamics of a harmonic oscillator if y_0 and σ_0 satisfy the relation $y_0\cos^2\sigma_0 - (mwQ_0 - y_0\cos\sigma_0\sin\sigma_0)\tan\sigma_0 = P_0$. In other words, in order to obtain the familiar form, $Q = Q_0\cos(w\tau) + P_0\sin(w\tau)/(mw)$ and $P = -mwQ_0\sin(w\tau) + P_0\cos(w\tau)$, we fix y_0 as $y_0 = P_0 + mwQ_0\tan\sigma_0$. The value $\sigma_0 = 0$ is a good choice for the parameter σ_0, because in this case $y_0 = P_0$, $\Omega(t_0) = 0$ and $S(Q, P, t_0) = P_0 Q$.

The Schrödinger equation (60) takes the form

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_+(Q,P,t) = \left[(y(t)/m - \Omega(t)Q)\breve{P} + m\omega^2 Q\breve{Q} + \frac{1}{2m}\breve{P}^2 + \frac{1}{2}m\omega^2\breve{Q}^2\right]\psi_+(Q,P,t). \tag{96}$$

To solve this equation is advantageous and instructive to use the relation $|\psi(t)\rangle = \exp\left(-(\mathrm{i}/\hbar)(t-t_0)\widehat{H}\right)|\psi(t_0)\rangle$ into (59). Next, we make use of (2*a*) and transformations [29]

$$\exp\left((\mathrm{i}/\hbar)\tau\hat{H}\right)\widehat{q}\exp\left(-(\mathrm{i}/\hbar)\tau\hat{H}\right) = \cos(\omega\tau)\hat{q} + \sin(\omega\tau)\hat{p}/(m\omega) \tag{97a}$$

$$\exp\left((\mathrm{i}/\hbar)\tau\hat{H}\right)\widehat{p}\exp\left(-(\mathrm{i}/\hbar)\tau\hat{H}\right) = \cos(\omega\tau)\hat{p} - m\omega\sin(\omega\tau)\hat{q} \tag{97b}$$

to rewrite (59) as

$$\psi_+(Q,P,t) = \exp\left(-\frac{\mathrm{i}}{\hbar}S(Q,P,t)\right)\langle 0|\exp\left(-\frac{\mathrm{i}}{\hbar}\tau\hat{H}\right)\widehat{D}\left(Q',P'\right)\widehat{D}\left(Q'',P''\right)|\psi(t_0)\rangle \tag{97c}$$

where $Q' = P\sin(\omega\tau)/(m\omega)$, $P' = -P\cos(\omega\tau)$, $Q'' = -Q\cos(\omega\tau)$ and $P'' = -Qm\omega\sin(\omega\tau)$.

Using (63) and (1*a*) and (1*b*), and noting that $\hat{H}|0\rangle = (\hbar\omega/2)|0\rangle$, we obtain

$$\psi_+(Q,P,t) = \exp\left(-\frac{\mathrm{i}\omega\tau}{2}\right)\exp\left(-\frac{\mathrm{i}}{\hbar}S(Q,P,t)\right)\exp\left(\frac{1}{2}\frac{\mathrm{i}}{\hbar}QP\right) \times \underset{-}{w}\left(Q_0, \frac{1}{2}P_0\right)\langle 0|\underset{+}{\hat{D}}(Q_0,P_0)|\psi(t_0)\rangle \tag{98a}$$

where $Q_0 = Q\cos(\omega\tau) - P\sin(\omega\tau)/(m\omega)$ and $P_0 = P\cos(\omega\tau) + m\omega Q\sin(\omega\tau)$. With (64) we obtain $\langle 0|\underset{+}{\hat{D}}(Q_0, P_0)\,|\psi(t_0)\rangle = \exp\left(\mathrm{i}S(Q_0, P_0, t_0)/\hbar\right)\psi_+(Q_0, P_0, t_0)$, where $S(Q_0, P_0, t_0) = P_0Q_0$ (with $\sigma_0 = 0$), and $\psi_+(Q_0, P_0, t_0)$ is the initial phase-space state. Finally, from (98*a*) we obtain

$$\psi_+(Q, P, t) = \exp\left(-\frac{\mathrm{i}\omega\tau}{2}\right)\exp\left(\frac{1}{2}\frac{\mathrm{i}}{\hbar}\,(QP - Q_0P_0)\right) \times \exp\left(-\frac{\mathrm{i}}{\hbar}\,(S(Q, P, t) - S(Q_0, P_0, t_0))\right)\psi_+(Q_0, P_0, t_0). \tag{98b}$$

Note that in this example $S(Q, P, t)$ is a complex function (see (92) and (93*c*)). The wavefunction $\psi_+(Q, P, t)$ depends periodically on time and each phase-space point of the initial state $\psi(Q_0, P_0, t_0)$ is propagated along the corresponding classical trajectory. However, the whole Husimi function stays in its original phase-space position because there is not a force applied to the particle.

7.4. Visualization of phase-space dynamics

We can gain a better understanding of the behaviour of the phase-space wavefunction $\psi_+(Q, P, t)$ by considering a specific initial state. Let $|\psi(t_0)\rangle$ be a coherent state $|\kappa_0\overline{q} + \mathrm{i}\chi_0\overline{p}\rangle$, with mean momentum $\overline{p}$ and coordinate $\overline{q}$. With the help of (57*a*) and (57*b*), (1*a*) and (1*b*) and (63), we obtain for the initial phase-space wavefunctions $\psi_\pm(Q, P, t_0)$ the relations

$$\begin{aligned}\exp\left(\pm\frac{\mathrm{i}}{\hbar}S(Q, P, t_0)\right)\psi_\pm(Q, P, t_0) &= \langle 0|\underset{\pm}{\hat{D}}(Q, P)\,|\kappa_0\overline{q} + \mathrm{i}\chi_0\overline{p}\rangle \\ &= \exp\left(\pm\frac{\mathrm{i}}{2\hbar}QP\right) \\ &\quad\times\exp\left(\frac{\mathrm{i}}{2\hbar}(Q\overline{p} - P\overline{q})\right)\left[(\pi\hbar)^{1/2}M\,(Q - \overline{q})\,\tilde{M}\,(P - \overline{p})\right]^{1/2}\end{aligned} \tag{99a}$$

where $S(Q, P, t_0) = P_0Q$ for the three examples presented in this section (set $\sigma_0 = 0$ in the case of Harmonic oscillator).

We may also use (67), (70), and (99*a*), to find the initial position-space and momentum-space wavefunctions

$$\varphi(Q, t_0) := \langle Q|\psi(t_0)\rangle = M\,(Q - \bar{q})\exp\left(\frac{\mathrm{i}}{\hbar}Q\bar{p}\right)\exp\left(-\frac{\mathrm{i}}{2\hbar}\bar{q}\bar{p}\right) \tag{99b}$$

$$\tilde{\varphi}(P, t_0) := \langle P|\psi(t_0)\rangle = \tilde{M}\,(P - \bar{p})\exp\left(-\frac{\mathrm{i}}{\hbar}P\bar{q}\right)\exp\left(\frac{\mathrm{i}}{2\hbar}\bar{q}\bar{p}\right). \tag{99c}$$

In the case of a particle in a *time-dependent linear potential*, the phase-space wavefunction (86*b*) can be written as

$$\psi_+(Q, P, t) = \frac{1}{\sqrt{\sqrt{\pi}q_0\,(1 + 2\sigma(t))}} \times\int_{-\infty}^{\infty}\mathrm{d}Q'\exp\left(-\frac{\left(Q' - Q_0\right)^2}{2q_0^2(1 + 2\sigma(t))}\right)\exp\left(-\frac{\mathrm{i}}{\hbar}P_0Q'\right)\psi(Q', t_0). \tag{100}$$

In particular, if the initial state $\psi(Q', t_0)$ is the coherent state $\varphi(Q', t_0)$, then the phase-space state at the later time t is

$$\psi_+(Q, P, t) = \frac{1}{\sqrt{1 + \sigma(t)}}\exp\left(A(Q, P, t)\right) \tag{101a}$$

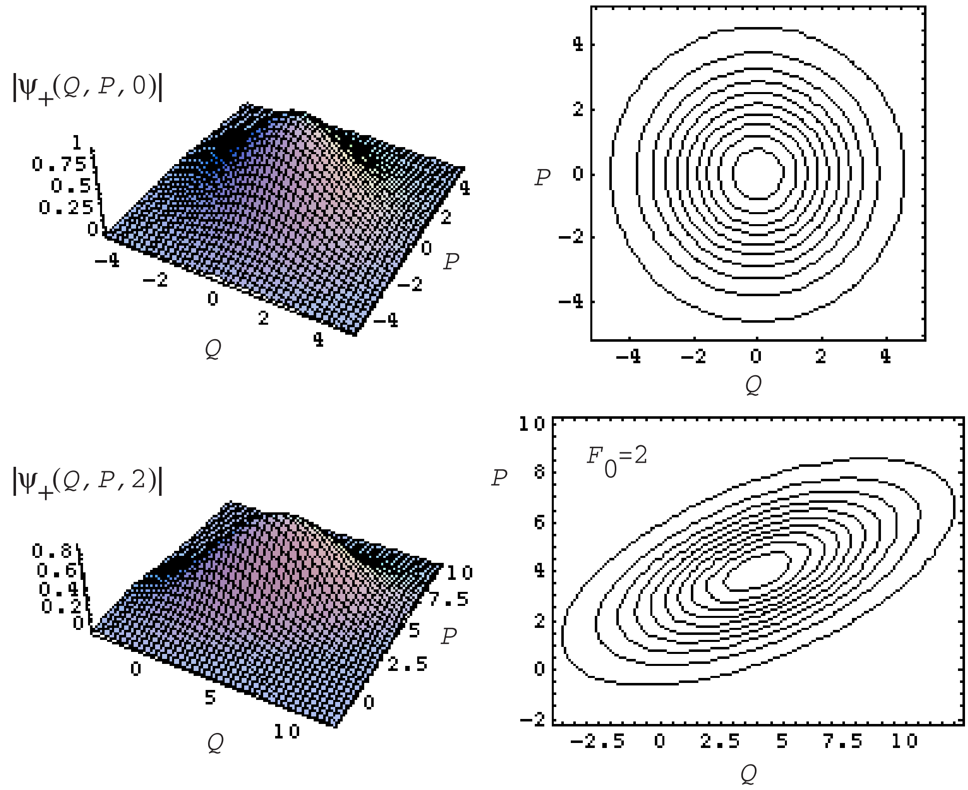

Figure 1. Phase-space evolution of an initial coherent state which is launched in a time-independent linear potential. The figures show three-dimensional surface plots of $|\psi_+(Q, P, t)|$, an its contours, at initial time $t_0 = 0$ and at a later time $t = 2$. We set $m = 1$, $q_0 = p_0 = 1$, $\overline{q} = \overline{p} = 0$, $F(t) = F_0 = 2$.

where

$$A(Q, P, t) := -\frac{1 + 2\sigma(t)}{4(1 + \sigma(t))}\left(\frac{P_0 - \overline{p}}{p_0}\right)^2 - \frac{1}{4(1 + \sigma(t))}\left(\frac{Q_0 - \overline{q}}{q_0}\right)^2$$
$$-\frac{\mathrm{i}}{2\hbar(1 + \sigma(t))}(P_0 - \overline{p})\,[Q_0 + (1 + 2\sigma(t))\overline{q}] - \frac{\mathrm{i}}{2\hbar}\overline{q}\,\overline{p} \qquad (101b)$$

and $q_0^2\sigma(t) := \mathrm{i}\hbar\,(t - t_0)\,/(2m)$, $Q_0 = Q - (t - t_0)P/m - \chi(t)$ and $P_0 = P - g(t)$. In the case of a *free particle*, $\chi(t) = 0$ and $g(t) = 0$.

Evolution of the quantum-mechanical state is described either by the function $\psi_+(Q, P, t)$ of Q, P and t, as seen by an observer at fixed phase-space points (Q, P), or by the wavefunction $\overline{\psi}_+(Q_0, P_0, t) := \psi_+(Q(t), P(t), t)$ as seen by an observer moving along a classical phase-space trajectory $(Q(t), P(t))$ with initial condition (Q_0, P_0) at time t_0. Using $S(Q, P, t_0) = P_0 Q$, $Q(t_0) = Q_0$, and $P(t_0) = P_0$, we see that in the limit $t \to t_0$, equations (101*a*) and (101*b*) reduce to the initial state (99*a*). We may also verify by substitution that (101*a*) and (101*b*) are a solution of (85).

Consider the case of an initial coherent state with $\overline{q} = \overline{p} = 0$, and apply to the particle a force $F(t) = F_0$. Figure 1 shows three-dimensional surface plots of $|\psi_+(Q, P, t)|$, and its contours, at initial time $t_0 = 0$ and at a later time $t = 2$. The function $|\psi_+(Q, P, t)|$ takes on

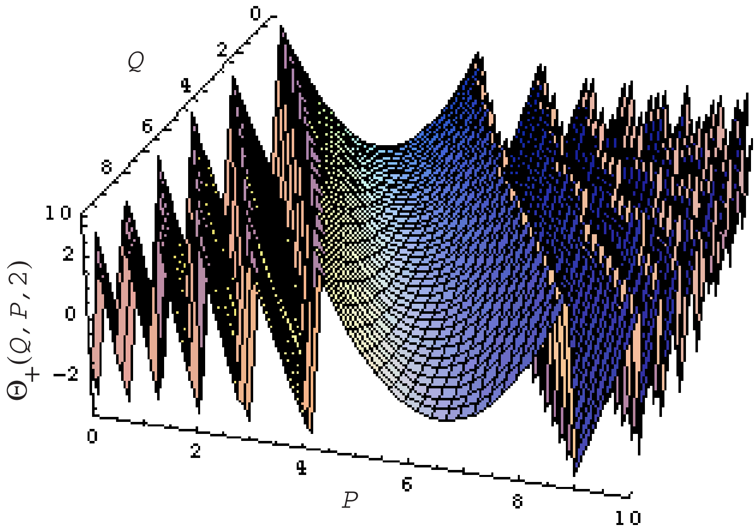

Figure 2. Phase-space evolution of an initial coherent state which is launched in a time-independent linear potential. This figure show a three-dimensional surface plot of the argument $\Theta(Q, P, t)$ of the wavefunction $\psi_+(Q, P, t)$ at time $t = 2$. We set $m = 1$, $q_0 = p_0 = 1$, $\overline{q} = \overline{p} = 0$, $F(t) = F_0 = 2$.

the same value along each contour. Figure 2 shows the behaviour of the argument $\Theta(Q, P, t)$ of the wavefunction $\psi_+(Q, P, t)$, at time $t = 2$.

The time development of $\psi(Q, P, t)$ is determined by three contributions:

(a) the initial state $\psi(Q, P, t_0)$;

(b) a classical evolution of each phase-space point of the initial state; and

(c) a quantum effect described by the complex function $\sigma(t)$.

The departure of the classical behaviour is due to the time dependence of $\sigma(t)$. Note that the centre of the Husimi function moves in phase space because there is an applied external force, F_0.

8. A suitable choice of $S(Q, P, t)$ in the case of a standard Hamiltonian

In this section we consider a particular case of a Hamiltonian (30), namely $H\big(\hat{q}, \hat{p}, t\big) = T(\hat{p}, t) + V(\hat{q}, t)$. Functions $T(p, t)$ and $V(q, t)$ have Taylor expansions with coefficients $T_n(t)$ and $V_n(t)$. By using (31), expansions for $(q + \breve{Q})^n$ and $(p + \breve{P})^n$, see (21), we obtain

$$\underset{+}{\mathcal{H}}\left(\breve{Q}_+, \breve{P}_+, t\right) = \sum_{n=0}^{\infty} T_n(t) \sum_{k=0}^{n} p_{nk} \breve{P}^k + \sum_{n=0}^{\infty} V_n(t) \sum_{k=0}^{n} q_{nk} \breve{Q}^k \tag{102}$$

where $q_{n,m}$ and $p_{n,m}$ are functions of (Q, P, t) (see section 4.1, appendix B, and note that $X \to q$ and $Y \to p$).

By comparison of (102) with (37), we obtain

$$K(Q,P,t) = \sum_{n=0}^{\infty} T_n(t) p_{n,0} + \sum_{n=0}^{\infty} V_n(t) q_{n,0} \tag{103a}$$

$$K_{10}(Q,P,t) = \sum_{n=0}^{\infty} T_{n+1}(t) p_{n+1,1} \tag{103b}$$

$$K_{01}(Q,P,t) = \sum_{n=0}^{\infty} V_{n+1}(t) q_{n+1,1} \tag{103c}$$

$$\underset{+}{\mathcal{D}}\left(Q,P,\check{Q},\check{P},t\right) = \sum_{k=2}^{\infty}\left(\sum_{n=0}^{\infty} T_{n+k}(t) p_{n+k,k}\right)\check{P}^k + \sum_{k=2}^{\infty}\left(\sum_{n=0}^{\infty} V_{n+k}(t) q_{n+k,k}\right)\check{Q}^k. \tag{103d}$$

In the following we restrict considerations to *standard* kinetic energy for an f-dimensional system, $T(\hat{p},t) = \hat{p}^2/(2m)$. In this case, $T_0 = T_1 = 0$, $T_2 = 1/(2m)$ and $T_n = 0$, for $n \geqslant 3$.

We may now seek a solution of the Hamilton–Jacobi equation (45) and generalized Hamilton equations (44) in terms of a P-independent function $S(Q,P,t) = S(Q,Y_0,t)$, where Y_0 denotes a set of parameters such as the initial impulse P_0 and the coordinate Q_0 (see the examples in section 7). As a consequence of this ansatz, we get (see (B2) and the last paragraph of appendix B)

$$p_{20} = p^2 + (\check{P}p) = p^2 - \mathrm{i}\hbar\frac{\partial^2 S}{\partial Q^2} \qquad p = \frac{\partial S}{\partial Q} \tag{104a}$$

$$q_{nk} = \binom{n}{k} Q^{n-k} \qquad \text{for} \quad n = 1,2,3,\ldots, \quad \text{and} \quad 0 \leqslant k \leqslant n. \tag{104b}$$

Thus, equations (103*a*)–(103*d*) become

$$K(Q,P,t) = \frac{1}{2m}\left(p^2 - \mathrm{i}\hbar\frac{\partial^2 S}{\partial Q^2}\right) + V(Q,t) \tag{105a}$$

$$K_{10}(Q,P,t) = \frac{p}{m}, K_{01}(Q,P,t) = \frac{\partial}{\partial Q}V(Q,t) \tag{105b}$$

$$\underset{+}{\mathcal{D}}\left(Q,P,\check{Q},\check{P},t\right) = \frac{1}{2m}\check{P}^2 + \sum_{k=2}^{\infty} V_k(Q,t)\check{Q}^k \tag{105c}$$

where, according to notation (17*b*), $k!V_k(Q,t) := (\partial/\partial Q)^k V(Q,t)$. Corresponding to these results, the quantum Hamilton–Jacobi equation (45) and generalized Hamilton equations (44) can be expressed in the form

$$\frac{\partial}{\partial t}S(Q,Y_0,t) + \frac{1}{2m}\left[S_1^2(Q,Y_0,t) - \mathrm{i}\hbar 2S_2(Q,Y_0,t)\right] + V(Q,t) = 0 \tag{106}$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{S_1(Q,Y_0,t)}{m} \qquad \frac{\mathrm{d}P}{\mathrm{d}t} = -V_1(Q,t) \tag{107}$$

with

$$n!S_n(Q,Y_0,t) := (\partial/\partial Q)^n S(Q,Y_0,t) = S^{(n)}(Q,Y_0,t)$$

and

$$S_0(Q,Y_0,t) := S(Q,Y_0,t).$$

In the limit $\hbar \to 0$ equation (106) reduces to the classical Hamilton–Jacobi equation. The original kinetic energy $p^2/(2m)$ is modified by an additional contribution $-\mathrm{i}\hbar S_2(Q,Y_0,t)/m$, which is a classical effect due to the quantum kinetic energy $\hat{p}^2/(2m)$. It is encouraging that equation (106) is identical to the so-called quantum Hamilton–Jacobi equation obtained in [47] by using the idea of quantum canonical transformations and generating functions.

A solution of generalized Hamilton equations (107) also becomes a solution of Hamilton's canonical equations, if the function $S(Q, Y_0, t)$ enables us to express P as a function of the variables Q, t, and parameters Y_0, by the relation $P = S_1(Q, Y_0, t)$. For this purpose, it is a *necessary requirement* to fix the initial condition for equation (106) as $S(Q, P_0, t_0) = P_0 Q$, so that $P(t_0) = P_0$. In this way, in the examples of section 7, we obtain the standard classical dynamics from the generalized Hamilton equations.

On the other hand, we know that the Hamilton–Jacobi equation (HJE) leads to canonical Hamilton equations as its bicharacteristic system [48]. Since, quantum HJE (106) differs from standard HJE by the term $-\mathrm{i}\hbar 2S_2(Q, Y_0, t)$, quantum effects may play a significant role in the generalized classical dynamics of the system described by (107).

Finally, the phase-space Schrödinger equation (60) can be written in the form

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_+(Q, P, t) = \left[\frac{P}{m}\breve{P} + \frac{1}{2m}\breve{P}^2 + \sum_{k=1}^{\infty} V_k(Q, t)\breve{Q}^k\right]\psi_+(Q, P, t). \qquad (108)$$

The quantity $P = S_1(Q, P_0, t)$ is the momentum of the classical particle that follows a phase-space trajectory connecting the points (Q_0, P_0) and (Q, P). Relation (108) is an infinite-order partial differential equation if $V(Q, t)$ does not take the form of a finite polynomial in Q.

To end this section let us note that a quantum system can also be described by the Wigner distribution function $W(Q, P, t)$ with time evolution ruled by [1]

$$\mathrm{i}\hbar\frac{\partial}{\partial t}W(Q, P, t) = \left[\frac{P}{m}\breve{P} + \sum_{k=0}^{\infty}\frac{1}{2^{2k}}V_{2k+1}(Q, t)\breve{Q}^{2k+1}\right]W(Q, P, t). \qquad (109)$$

Wigner function $W(Q, P, t)$ is real for all Q and P, and it is a bilinear form of the wavefunction, while $\psi_+(Q, P, t)$ in (108) is complex valued and it is a phase-space probability amplitude (see (57*a*)). If integrates over P, $W(Q, P, t)$ gives the proper probabilities for the different values of Q, and similarly with $P \leftrightarrow Q$. Integration of $\psi_+(Q, P, t)$ over P or Q gives the wavefunction $\psi(Q, t)$ or $\tilde{\psi}(P, t)$, and its derivatives with respect to Q or P (see (67) and (70)). To deal with (109), Lee and Scully [49] introduce an iteration scheme with an effective potential $V_{\mathrm{eff}}(Q, P, t)$ and the corresponding modified Hamilton equations, to get Wigner trajectories along which each phase-space point (Q, P) of the Wigner distribution moves. In the present treatment, equations (106) and (107) define the classical dynamics autonomously, while the quantum dynamics described by (108) is driven by the solution $S(Q, P_0, t)$ of the quantum Hamilton–Jacobi equation (106) and the solution $(Q(t), P(t))$ of the generalized Hamilton equations (107).

9. Discussion and concluding remarks

To begin these final remarks, let us summarize the main steps of the method: in the first part of the paper, we introduced a general 'classicalization' procedure. For an arbitrary Hilbert-space operator $\underset{\pm}{F}(\hat{q}, \hat{p}, t)$ and an arbitrary smooth phase-space function $S(Q, P, t)$, we found a phase-space representation $\underset{\pm}{\mathcal{F}}\left(\breve{Q}_\pm[S], \breve{P}_\pm[S], t\right)$, see (8*a*) and (8*b*) and (16). Then, in the second part of the paper, we dealt mainly with the time-dependent Schrödinger equation for a general quantum system with Hamiltonian $H(\hat{q}, \hat{p}, t)$. The steps were:

(a) We constructed an operator $\underset{+}{\mathcal{H}}\left(\breve{Q}_+, \breve{P}_+, t\right)$ and obtained a generalized Hamiltonian $K(Q, P, t)$ and functions $K_{10}(Q, P, t)$ and $K_{01}(Q, P, t)$. We solved the generalized Hamilton equations (44). Thus, we obtained a phase-space trajectory $(Q(t), P(t))$ that connects the initial state (Q_0, P_0) to the state $(Q(t), P(t))$ at time t.
(b) As quantum time evolution can be described from two equivalent points of view (travelling and fixed observers), we solved the associated quantum-mechanical equation of motion using the most convenient one, e.g. (49*a*) and (49*b*).
(c) Finally, using (61*b*), we evaluated expectation values and other quantum-mechanical quantities.

For an arbitrary quantum-mechanical system described by the Hamilton operator $H(\hat{q}, \hat{p}, t)$, we have found a classical counterpart identified by the generalized Hamiltonian $K(Q, P, t)$. The classical system, however, does not obey standard Hamilton equations, but generalized Hamilton equations (44). This result could indicate that classical mechanics is not strictly the limiting case of quantum mechanics, but a separate and different theory, as suggested by Casati and Chirikov [30].

Present results could also constitute an alternative for studying the quantum-classical association, which traditionally has been analysed from different perspectives, for example: the semiclassical trace formula [31] and its extensions [32, 33], the standard semiclassical dynamics of Gaussian wavepackets [18], the Gaussian semiquantal dynamics [14–17], the Wigner–Liouville formalism [1, 4, 34, 35], the de Broglie–Bohm (Hamilton–Jacobi) formulation [19, 36].

Many papers maintain that phase-space trajectories cannot be defined for quantum systems, because the uncertainty principle makes simultaneous specification of position and momentum impossible. The herein introduced phase-space trajectories originate from applying the transformation (41) to the Schrödinger equation (41) includes position-$Q(t)$ and momentum-$P(t)$ free parameters that were chosen suitably afterwards (generalized Hamilton equations (44)). We have refrained from using representation theory and have treated $\hat{q}$ and $\hat{p}$ as mere non-commuting Hilbert parameters, so that the uncertainty principle remains unscathed. This principle, which is actually a theorem about uncertainties,

$$\Delta q = \sqrt{\langle\psi(t)|\left(\hat{q} - \langle\hat{q}\rangle\right)^2|\psi(t)\rangle} \qquad \Delta p = \sqrt{\langle\psi(t)|\left(\hat{p} - \langle\hat{p}\rangle\right)^2|\psi(t)\rangle}$$

does not impose any *a priori* restriction on the free parameters $(Q(t), P(t))$. Thus, applying this method, we are able to incorporate the classical concept of a phase-space trajectory into quantum dynamics, despite the uncertainty principle. So, quantum evolution is partially driven by classical evolution and no contradiction with the uncertainty principle exists.

Evolution of quantum-mechanical states can be described either by the wavefunction $\psi_+(Q, P, t)$ at a fixed phase-space point (Q, P), or by the wavefunction $\psi_+(Q(t), P(t), t)$ along the classical phase-space trajectory $(Q(t), P(t))$. Although this structure resembles classical statistical mechanics, here we have wavefunctions $\psi_+(Q, P, t)$ parametrized by (Q, P) phase-space points instead of probability distributions, $\rho(Q, P, t) \geqslant 0$. Nonetheless, the Husimi function $\rho(Q, P, t)$ can be directly obtained from $\psi_+(Q, P, t)$ as indicated after equation (58). In this paper we have obtained the motion equation for the amplitude of probability $\psi_+(Q, P, t)$, instead of the equation for the time development of $\rho(Q, P, t)$ [37].

The quantum distributions method (Wigner, Husimi, etc) provides a means to determine the quantum-mechanical averages in terms of phase-space integration over c-number variables, in a form quite similar to that applied to evaluate classical averages in statistical mechanics [4]. This requires the introduction of a mapping between Hilbert operators $F(\hat{q}, \hat{p})$ and classical phase-space function $F(Q, P)$, which unfortunately is not well defined in some cases. In this paper, formulae (61*b*) is similar to the standard quantum-mechanical procedures for evaluating quantum-mechanical averages, but in our case we use phase-space amplitudes. Note that, with the help of (10*a*) and (15), we obtain

$$\langle\varphi(t)|\,\underset{+}{\hat{D}}(Q,P)\,\underset{+}{F}(\widehat{q},\widehat{p},t)\,|\psi(t)\rangle = \exp\left(+\frac{\mathrm{i}}{\hbar}S(Q,P,t)\right)\underset{+}{\mathcal{F}}\left(\breve{Q}_+,\breve{P}_+,t\right)$$
$$\times\exp\left(-\frac{\mathrm{i}}{\hbar}S(Q,P,t)\right)\langle\varphi(t)|\,\underset{+}{\hat{D}}(Q,P)\,|\psi(t)\rangle. \tag{110}$$

Thus, the matrix elements of an arbitrary operator $\underset{+}{F}(\widehat{q},\widehat{p},t)$ can also be obtained by using the above equation in the first instance and then by setting $Q = P = 0$.

Let us finally note that the Husimi function has been used in a number of studies of quantum dynamical systems [38, 39], in particular, in a systematic searching for scars [40]. As expressed by Heller [41], the scars manifest themselves as an enhanced probability in phase space, as measured by overlap with coherent states placed on the periodic orbits. This also translates into an enhanced probability in coordinate space along the periodic orbits. In measuring scars, there are many methods [42–44], [45, and references therein], including the coherent state projection [46]. Note that the coherent state function contains information about the phase of the wavefunction and is hence a more fundamental object than the Husimi function (94).

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Appendix A

In this appendix, we outline the derivation of equations (10*a*), (10*b*) and (12*a*), (12*b*). We differentiate expressions (7*a*) and (7*b*) with operators $\pm\mathrm{i}\hbar\partial/\partial a$, and then putting $a = 0$, we get ($k = 0, 1, 2, \ldots$)

$$\overset{2k}{F}\left(\breve{P}\right)\underset{+}{\hat{D}}(Q,P) = \underset{+}{\hat{D}}(Q,P)\overset{2k}{F}\left(\hat{p}\right) \tag{A1}$$

$$\overset{2k+1}{F}\left(\breve{Q}\right)\underset{-}{\hat{D}}(Q,P) = \underset{-}{\hat{D}}(Q,P)\overset{2k+1}{F}\left(\hat{q}\right) \tag{A2}$$

$$\overset{2k}{F}\left(\breve{P}\right)\underset{-}{\hat{D}}(Q,P) = \overset{2k}{F}\left(\hat{p}\right)\underset{-}{\hat{D}}(Q,P) \tag{A3}$$

$$\overset{2k+1}{F}\left(\breve{Q}\right)\underset{+}{\hat{D}}(Q,P) = \overset{2k+1}{F}\left(\hat{q}\right)\underset{+}{\hat{D}}(Q,P). \tag{A4}$$

Since (1*b*) tells us that $\underset{+}{\hat{D}}(Q,P) = \underset{+}{w}(Q,P)\,\underset{-}{\hat{D}}(Q,P)$, it is possible to rewrite expressions (A1)–(A4) as (11*a*)–(11*d*), where we use the notation (12*c*).

Now, our task is to verify that the Hilbert-space operator (8*a*) is related to phase-space operator (12*a*) through relation (10*a*). We begin by considering equations (11*a*) and (11*b*), with $k = 0$:

$$\underset{-}{\overset{0}{F}}(Q, P, \breve{P})\, \underset{+}{\hat{D}}(Q, P) = \underset{-}{\hat{D}}(Q, P)\, \overset{0}{F}(\hat{p}) \tag{A5}$$

$$\underset{+}{\overset{1}{F}}(Q, P, \breve{Q})\, \underset{-}{\hat{D}}(Q, P) = \underset{+}{\hat{D}}(Q, P)\, \overset{1}{F}(\hat{q}). \tag{A6}$$

Then, right multiplication of (A6) by $\overset{0}{F}\left(\hat{p}\right)$ leads to

$$\underset{+}{\overset{1}{F}}(Q, P, \breve{Q})\, \underset{-}{\hat{D}}(Q, P)\, \overset{0}{F}\left(\hat{p}\right) = \underset{+}{\hat{D}}(Q, P)\, \overset{1}{F}(\hat{q})\, \overset{0}{F}\left(\hat{p}\right).$$

Now, by using (A5), we obtain

$$\underset{+}{\overset{1}{F}}(Q, P, \breve{Q})\, \underset{-}{\overset{0}{F}}(Q, P, \breve{P})\, \underset{+}{\hat{D}}(Q, P) = \underset{+}{\hat{D}}(Q, P)\, \overset{1}{F}(\hat{q})\, \overset{0}{F}\left(\hat{p}\right). \tag{A7}$$

Proceeding similarly to the previous step, we consider (11*a*) with $k = 1$, and by right multiplication with $\overset{1}{F}(\hat{q})\, \overset{0}{F}\left(\hat{p}\right)$ and using (A7), we obtain

$$\underset{-}{\overset{2}{F}}(Q, P, \breve{P})\, \underset{+}{\overset{1}{F}}(Q, P, \breve{Q})\, \underset{-}{\overset{0}{F}}(Q, P, \breve{P})\, \underset{+}{\hat{D}}(Q, P) = \underset{-}{\hat{D}}(Q, P)\, \overset{2}{F}(\hat{p})\, \overset{1}{F}(\hat{q})\, \overset{0}{F}\left(\hat{p}\right). \tag{A8}$$

Now, by using (1*b*), as $\underset{-}{\hat{D}}(Q, P) = \underset{-}{w}(Q, P)\, \underset{+}{\hat{D}}(Q, P)$, equation (A8) can be written as

$$\underset{+}{w}(Q, P)\, \underset{-}{\overset{2}{F}}(Q, P, \breve{P})\, \underset{+}{\overset{1}{F}}(Q, P, \breve{Q})\, \underset{-}{\overset{0}{F}}(Q, P, \breve{P})\, \underset{+}{\hat{D}}(Q, P) = \underset{+}{\hat{D}}(Q, P)\, \overset{2}{F}(\hat{p})\, \overset{1}{F}(\hat{q})\, \overset{0}{F}\left(\hat{p}\right). \tag{A9}$$

Compare this equation with (10*a*), (8*a*), (12*a*) for N even ($N = 2$).

In general, in the above way, we obtain a sequence of equations. They can be written as the intertwining relation (10*a*), where (8*a*) and (12*a*) define suitable Hilbert-space and phase-space operators. A similar procedure with equations (11*c*) and (11*d*) leads to the intertwining relation (10*b*) with definitions (8*b*) and (12*b*).

Appendix B

Consider expansion (21) of phase-space operator $(X + \breve{Q})^n$. The coefficients $X_{n\mu}(Q, P, t)$ are determined by using (23) and (22). We obtain

$$X_{10} = X \tag{B1}$$

$$X_{20} = X^2 + (\breve{Q}X) \tag{B2}$$

$$X_{30} = X^3 + 3(\breve{Q}X)X + (\breve{Q}^2X) \tag{B3}$$

$$X_{40} = X^4 + 6(\breve{Q}X)X^2 + 4(\breve{Q}^2X)X + 3(\breve{Q}X)^2 + (\breve{Q}^3X) \tag{B4}$$

$$\begin{aligned} X_{50} = {} & X^5 + 10(\breve{Q}X)X^3 + 10(\breve{Q}^2X)X^2 + 15(\breve{Q}X)^2X \\ & + 5(\breve{Q}^3X)X + 10(\breve{Q}X)(\breve{Q}^2X) + (\breve{Q}^4X) \end{aligned} \tag{B5}$$

$$\begin{aligned} X_{60} = {} & X^6 + 15(\breve{Q}X)X^4 + 20(\breve{Q}^2X)X^3 + 45(\breve{Q}X)^2X^2 \\ & + 15(\breve{Q}^3X)X^2 + 60(\breve{Q}X)(\breve{Q}^2X)X + 6(\breve{Q}^4X)X \\ & + 15(\breve{Q}X)^3 + 10(\breve{Q}^2X)^2 + 15(\breve{Q}X)(\breve{Q}^3X) + (\breve{Q}^5X). \end{aligned} \tag{B6}$$

$$X_{11} = 1 \tag{B7}$$
$$X_{21} = 2X \tag{B8}$$
$$X_{31} = 3X^2 + 3(\breve{Q}X) \tag{B9}$$
$$X_{41} = 4X^3 + 12(\breve{Q}X)X + 4(\breve{Q}^2X) \tag{B10}$$
$$X_{51} = 5X^4 + 30(\breve{Q}X)X^2 + 20(\breve{Q}^2X)X + 15(\breve{Q}X)^2 + 5(\breve{Q}^3X) \tag{B11}$$
$$X_{61} = 6X^5 + 60(\breve{Q}X)X^3 + 60(\breve{Q}^2X)X^2 + 90(\breve{Q}X)^2X + 30(\breve{Q}^3X)X + 60(\breve{Q}X)(\breve{Q}^2X) + 6(\breve{Q}^4X). \tag{B12}$$

$$X_{22} = 1 \tag{B13}$$
$$X_{32} = 3X \tag{B14}$$
$$X_{42} = 6X^2 + 6(\breve{Q}X) \tag{B15}$$
$$X_{52} = 10X^3 + 30(\breve{Q}X)X + 10(\breve{Q}^2X) \tag{B16}$$
$$X_{62} = 15X^4 + 90(\breve{Q}X)X^2 + 60(\breve{Q}^2X)X + 45(\breve{Q}X)^2 + 15(\breve{Q}^3X). \tag{B17}$$

$$X_{33} = 1 \tag{B18}$$
$$X_{43} = 4X \tag{B19}$$
$$X_{53} = 10X^2 + 10(\breve{Q}X) \tag{B20}$$
$$X_{63} = 20X^3 + 60(\breve{Q}X)X + 20(\breve{Q}^2X). \tag{B21}$$

$$X_{44} = 1 \tag{B22}$$
$$X_{54} = 5X \tag{B23}$$
$$X_{64} = 15X^2 + 15(\breve{Q}X) \tag{B24}$$

$$X_{55} = 1 \tag{B25}$$
$$X_{65} = 6X. \tag{B26}$$

Note that $X_{n\mu}$ reduces to standard binomial coefficients, $\binom{n}{\mu}X^{n-\mu}$, in the case of a P-independent function X, that is, if $X = X(Q, t)$. In fact, in this case, we have $(\breve{Q}^{\mu}X) = 0$, for $\mu = 1, 2, \ldots$.

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